

Vanishing shear viscosity and boundary layers for plane magnetohydrodynamics flows

Wenshu Zhou^{1*} Xulong Qin² Chengyuan Qu¹

1. Department of Mathematics, Dalian Nationalities University, Dalian 116600, China

2. Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, China

Abstract

In this paper, we consider an initial-boundary problem for plane magnetohydrodynamics flows under the general condition on the heat conductivity κ that may depend on both the density ρ and the temperature θ and satisfies

$$\kappa(\rho, \theta) \geq \kappa_1(1 + \theta^q) \quad \text{with constants } \kappa_1 > 0 \text{ and } q > 0.$$

We prove the global existence of strong solutions for large initial data and justify the passage to the limit as the shear viscosity μ goes to zero. Furthermore, the value μ^α with any $0 < \alpha < 1/2$ is established for the boundary layer thickness.

Keywords. plane magnetohydrodynamics flows; global existence; vanishing shear viscosity; boundary layer.

2010 MSC. 35B40; 35B45; 76N10; 76N20; 76W05; 76X05.

1 Introduction

Magnetohydrodynamics (MHD) concerns the motion of conducting fluids in an electromagnetic field and has a very broad range of applications. The dynamic motion of the fluid and the magnetic field interact strongly on each other, so the hydrodynamic and electrodynamic effects are coupled, which make the problem considerably complicated. The plane MHD flows with constant longitudinal magnetic field, which are three-dimensional MHD flows uniform in the transverse direction, are governed by the following equations:

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + \left(\rho u^2 + p + \frac{1}{2} |\mathbf{b}|^2 \right)_x &= (\lambda u_x)_x, \\ (\rho \mathbf{w})_t + (\rho u \mathbf{w} - \mathbf{b})_x &= (\mu \mathbf{w}_x)_x, \\ \mathbf{b}_t + (u \mathbf{b} - \mathbf{w})_x &= (\nu \mathbf{b}_x)_x, \\ (\rho e)_t + (\rho u e)_x - (\kappa \theta_x)_x + p u_x &= \mathcal{Q}, \\ \mathcal{Q} &:= \lambda u_x^2 + \mu |\mathbf{w}_x|^2 + \nu |\mathbf{b}_x|^2, \end{aligned} \tag{1.1}$$

*Corresponding author.

E-mail address: mathpde@163.com (W.Zhou), qin_xulong@163.com (X. Qin); mathqcy@163.com (C. Qu)

where ρ denotes the density of the flow, θ the temperature, $u \in \mathbb{R}$ the longitudinal velocity, $\mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$ transverse velocity, $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$ transverse magnetic field, $p = p(\rho, \theta)$ the pressure, $e = e(\rho, \theta)$ the internal energy, and $\kappa = \kappa(\rho, \theta)$ the heat conductivity. The coefficients λ , μ and ν standing for the bulk viscosity, shear viscosity and the magnetic diffusivity, respectively, are assumed to be positive constants in this paper. We focus on the perfect gas with the equations of state:

$$p = \gamma \rho \theta, \quad e = c_v \theta, \quad (1.2)$$

where the constants $\gamma > 0$ and $c_v > 0$. Without loss of generality, we set $c_v = 1$.

We consider system (1.1) in the bounded domain $Q_T = \Omega \times (0, T)$ with $\Omega = (0, 1)$ subject to the following initial and boundary conditions:

$$\begin{cases} (\rho, u, \theta, \mathbf{w}, \mathbf{b})(x, 0) = (\rho_0, u_0, \theta_0, \mathbf{w}_0, \mathbf{b}_0)(x), \\ (u, \mathbf{b}, \theta_x)|_{x=0,1} = \mathbf{0}, \\ \mathbf{w}(0, t) = \mathbf{w}^-(t), \quad \mathbf{w}(1, t) = \mathbf{w}^+(t). \end{cases} \quad (1.3)$$

Because of physical importance, complexity, rich phenomenon, and mathematical challenges, the MHD problem has been extensively studied in many papers, see [1, 3–6, 15, 16, 18, 21, 23, 28, 30, 32, 33] and the references therein. In particular, if there is no magnetic effect, MHD reduces to the compressible Navier-Stokes equations, see for example [10, 19, 27, 29, 31, 34] and references therein for some mathematical studies. However, many fundamental problems for MHD are still open. For example, even though for the one dimensional compressible Navier-Stokes equations, there is a pioneer work by Kazhikhov and Shelukhin [20] on the global existence of strong solutions with large initial data, the corresponding problem for the MHD system with constant viscosity, heat conductivity and diffusivity coefficients remains unsolved. The reason is that the presence of the magnetic field and complex interaction with the hydrodynamic motion in the MHD flow of large oscillation cause serious difficulties.

The initial-boundary value problem (1.1)-(1.3) has fundamental importance in the studies on the MHD problem. In this paper, we investigate the global existence, zero shear viscosity limit, convergence rate and boundary layer effect of strong solutions for problem (1.1)-(1.3) with large initial data, where κ may depend both density and temperature such that $\kappa = \kappa(\rho, \theta)$ is twice continuous differential in $\mathbb{R}^+ \times \mathbb{R}^+$ and satisfies

$$\kappa(\rho, \theta) \geq \kappa_1(1 + \theta^q) \quad \text{with constants } \kappa_1 > 0 \text{ and } q > 0. \quad (1.4)$$

In kinetic theory of gas, the heat conductivity κ is a function of temperature θ and increases with θ in general (cf. [2, 36]). From experimental results for gases at very high temperatures (see [36]), the condition (1.4) seems reasonable when one considers a gas model that incorporates real gas effects that occur in high temperature (cf. [19]). In [19], one of the assumptions on κ is that there are constants $C_1, C_2 > 0$ such that the heat conductivity κ satisfies

$$C_1(1 + \theta^q) \leq \kappa(\rho, \theta) \leq C_2(1 + \theta^q), \quad \forall \rho, \theta > 0, \quad (1.5)$$

where $q \geq 2$, which implies that κ has a positive lower bound. This type of temperature dependence is also motivated by the physical fact that κ grows like θ^q with $q = 2.5$ for

important physical regimes and $q \in [4.5, 5.5]$ for molecular diffusion in gas (see [36]). The assumption (1.5) with $q > 0$ was also made in many papers (see for example [4, 7, 9, 17, 30] and references therein). Clearly, here we remove these assumptions on κ .

The well-posedness theory has been studied in many papers, some of which will be mentioned below. It was Vol'pert and Hudjaev [33] who first proved the existence and uniqueness of local smooth solutions. The global existence of smooth solutions with small initial data was established by Kawashima and Okada [18]. Under the technical condition on κ :

$$C^{-1}(1 + \theta^q) \leq \kappa(\rho, \theta) \equiv \kappa(\theta) \leq C(1 + \theta^q), \quad (1.6)$$

for $q \geq 2$, Chen and Wang [4] proved the existence, uniqueness and the Lipschitz continuous dependence of global strong solutions with large H^1 initial data. Similar results can be found in [3, 30] under the same technical condition as (1.6). The existence of global weak solutions was proved by Fan, Jiang and Nakamura [7] under the condition (1.6) for $q \geq 1$ or the condition $\kappa \equiv \kappa(\rho) \geq C/\rho$, while the uniqueness and the Lipschitz continuous dependence on the initial data of global weak solutions with the initial data in Lebesgue spaces were obtained by them in [8]. Very recently, the case $q > 0$ of condition (1.6) was treated by Fan, Huang and Li [9] where the existence and uniqueness of global solutions with large initial data and vacuum were shown. A similar result can be found in [14] by Hu and Ju. The uniqueness and continuous dependence of weak solutions for the Cauchy problem have been proved recently by Hoff and Tsyganov in [13]. In this paper, we show the global existence of strong solutions for problem (1.1)-(1.3) under the general condition (1.4), which extends some global existence results mentioned above.

The problem of small viscosity finds many applications, for example, in the boundary layer theory (cf. [26]). In this direction, some results on the Navier-Stokes equations can be referred to [11, 12, 17, 25, 27, 35] and references therein. The vanishing shear viscosity limit of the weak solution for problem (1.1)-(1.3) was studied by Fan, Jiang and Nakamura [7] under the condition (1.6) for $q \geq 1$ or $\kappa \equiv \kappa(\rho) \geq C/\rho$. As pointed out in [9], the result of [7] can be transplanted to the case $q > 0$ of the condition (1.6). In this paper, we justify the passage to the limit with more strong convergence of \mathbf{w} and \mathbf{b} under the general condition (1.4). Thus, we extend and improve some results mentioned above.

The boundary layer theory has been one of the fundamental and important issues in fluid dynamics since it was proposed by Prandtl in 1904. Frid and Shelukhin in [12] investigated the boundary layer effect of the compressible isentropic Navier-Stokes equations with cylindrical symmetry, and proved the existence of boundary layers of thickness $O(\mu^\alpha)$ ($0 < \alpha < 1/2$). Under the assumption on κ :

$$C^{-1}(1 + \theta^q) \leq \kappa(\rho, \theta) \leq C(1 + \theta^q), \quad |\kappa_\rho(\rho, \theta)| \leq C(1 + \theta^q), \quad q > 1, \quad (1.7)$$

Jiang and Zhang [17] studied the compressible nonisentropic Navier-Stokes equations with cylindrical symmetry, and proved that the thickness of boundary layer is of the order $O(\mu^\alpha)$ ($0 < \alpha < 1/2$). Recently, Jiang and Zhang's result is extended to the case of constant heat conductivity, see [25]. A similar result can be found in [35] by Yao, Zhang and Zhu. To the best of our knowledge, however, there is no corresponding results for the initial boundary problem (1.1)-(1.3). In this paper, the value μ^α ($0 < \alpha < 1/2$) is established for the boundary layer thickness of problem (1.1)-(1.3).

We introduce some notations. Let $k \geq 0$ be an integer, \mathcal{O} a domain of \mathbb{R}^n ($n \geq 1$) and $p \geq 1$. $W^{k,p}(\mathcal{O})$ and $W_0^{k,p}(\mathcal{O})$ denote the usual Sobolev spaces, $W^{0,p}(\mathcal{O}) = L^p(\mathcal{O})$. $C^k(\mathcal{O})$ and $C^k(\overline{\mathcal{O}})$ denote the spaces consisting of continuous derivatives up to order k in \mathcal{O} . For $0 < \alpha < 1$, $C^{k+\alpha}(\mathcal{O})$ (resp. $C^\alpha(\overline{\mathcal{O}})$) and $C^{k+\alpha,k+\alpha/2}(\mathcal{O})$ (resp. $C^{k+\alpha,k+\alpha/2}(\overline{\mathcal{O}})$) denote the Hölder spaces with the exponent α . $L^p(I, B)$ is the space of all strong measurable, p^{th} -power integrable (essentially bounded if $p = \infty$) functions from I to B , where $I \subset \mathbb{R}$ and B is a Banach space. For simplicity, we also use the notation $\|(f, g, \dots)\|_B^2 = \|f\|_B^2 + \|g\|_B^2 + \dots$ for functions f, g, \dots belonging to B equipped with a norm $\|\cdot\|_B$.

In what follows, we assume that the initial and boundary functions satisfy

$$\begin{aligned} \rho_0 > 0, \theta_0 > 0, \quad & \|(\rho_0^{-1}, \theta_0^{-1})\|_{C(\overline{\Omega})} < \infty, \quad \|(\mathbf{w}^-, \mathbf{w}^+)\|_{C^1([0,T])} < \infty, \\ (\rho_0, \mathbf{w}_0, \theta_0) & \in W^{1,2}(\Omega), \quad \mathbf{b}_0 \in W_0^{1,2}(\Omega), \quad u_0 \in W_0^{1,2}(\Omega) \cap W^{2,m}(\Omega) \text{ with } m \in (1, +\infty), \\ \mathbf{w}_0(0) &= \mathbf{w}^-(0), \quad \mathbf{w}_0(1) = \mathbf{w}^+(0). \end{aligned} \quad (1.8)$$

Now the results on the global existence, vanishing shear viscosity limit and convergence rate of strong solutions can be stated as follows.

Theorem 1.1. *Let (1.4) and (1.8) hold. Then*

(i) *For any fixed $\mu > 0$, there exists a unique strong solution $(\rho, u, \mathbf{w}, \mathbf{b}, \theta)$ for problem (1.1)–(1.3). Moreover, there exist some positive constants C independent of μ such that*

$$\begin{aligned} C^{-1} &\leq \rho, \theta \leq C, \quad \|(u, \mathbf{w}, \mathbf{b})\|_{L^\infty(Q_T)} \leq C, \\ \|(\rho_t, \rho_x, u_x, \mathbf{b}_x, \theta_x)\|_{L^\infty(0,T;L^2(\Omega))} &+ \|(u_t, \mathbf{b}_t, \theta_t, u_{xx}, \theta_{xx})\|_{L^2(Q_T)} \leq C, \\ \|\mathbf{w}_x\|_{L^\infty(0,T;L^1(\Omega))} &+ \|\mathbf{w}_t\|_{L^2(Q_T)} \leq C, \\ \mu^{1/4} \|\mathbf{w}_x\|_{L^\infty(0,T;L^2(\Omega))} &+ \mu^{3/4} \|\mathbf{w}_{xx}\|_{L^2(Q_T)} \leq C, \\ \|\sqrt{\omega} \mathbf{w}_x\|_{L^\infty(0,T;L^2(\Omega))} &+ \|(\sqrt{|u|} \mathbf{w}_x, \sqrt{\omega} \mathbf{b}_{xx})\|_{L^2(Q_T)} \leq C, \end{aligned} \quad (1.9)$$

where $\omega : [0, 1] \rightarrow [0, 1]$ is defined by

$$\omega(x) = \begin{cases} x, & 0 \leq x \leq 1/2, \\ 1 - x, & 1/2 \leq x \leq 1. \end{cases}$$

(ii) *There exist some functions $\bar{\rho}, \bar{u}, \bar{\mathbf{w}}, \bar{\mathbf{b}}$ and $\bar{\theta}$ in the class:*

$$\mathbb{F} : \begin{cases} \bar{\rho}, \bar{\theta} > 0, \quad (\bar{u}, \bar{\mathbf{b}})|_{x=0,1} = 0, \\ (\bar{\rho}, 1/\bar{\rho}, \bar{u}, \bar{\mathbf{w}}, \bar{\mathbf{b}}, \bar{\theta}, 1/\bar{\theta}) \in L^\infty(Q_T), \quad \bar{\mathbf{w}} \in L^\infty(0, T; W^{1,1}(\Omega)) \cap BV(Q_T), \\ (\bar{\rho}_t, \bar{\rho}_x, \bar{u}_x, \bar{\mathbf{b}}_x, \bar{\theta}_x) \in L^\infty(0, T; L^2(\Omega)), \quad (\bar{u}_x, \bar{\mathbf{b}}_x, \bar{\theta}_x) \in L^2(0, T; L^\infty(\Omega)), \\ (\bar{u}_t, \bar{\mathbf{w}}_t, \bar{\mathbf{b}}_t, \bar{\theta}_t, \bar{u}_{xx}, \bar{\theta}_{xx}) \in L^2(Q_T), \\ \sqrt{\omega} \bar{\mathbf{w}}_x \in L^\infty(0, T; L^2(\Omega)), \quad (\sqrt{|\bar{u}|} \bar{\mathbf{w}}_x, \sqrt{\omega} \bar{\mathbf{b}}_{xx}) \in L^2(Q_T), \end{cases}$$

such that, as $\mu \rightarrow 0$, $(\rho, u, \mathbf{w}, \mathbf{b}, \theta)$ converges in the following sense

$$\begin{aligned} (\rho, u, \mathbf{b}, \theta) &\rightarrow (\bar{\rho}, \bar{u}, \bar{\mathbf{b}}, \bar{\theta}) \text{ strongly in } C^\alpha(\bar{Q}_T), \quad \forall \alpha \in (0, 1/4), \\ (u_x, \theta_x) &\rightarrow (\bar{u}_x, \bar{\theta}_x) \text{ strongly in } L^{s_1}(Q_T), \quad \forall s_1 \in [1, 6), \\ \mathbf{b}_x &\rightarrow \bar{\mathbf{b}}_x \text{ strongly in } L^{s_2}(Q_T), \quad \forall s_2 \in [1, 4), \\ (\rho_t, \rho_x) &\rightharpoonup (\bar{\rho}_t, \bar{\rho}_x) \text{ weakly} - * \text{ in } L^\infty(0, T; L^2(\Omega)), \\ (u_t, \mathbf{b}_t, \theta_t, u_{xx}, \theta_{xx}) &\rightharpoonup (\bar{u}_t, \bar{\mathbf{b}}_t, \bar{\theta}_t, \bar{u}_{xx}, \bar{\theta}_{xx}) \text{ weakly in } L^2(Q_T), \\ \mathbf{b}_{xx} &\rightharpoonup \bar{\mathbf{b}}_{xx} \text{ weakly in } L^2((a + \delta, b - \delta) \times (0, T)), \quad \forall \delta \in (0, (b - a)/2), \end{aligned}$$

and

$$\begin{aligned} \mathbf{w} &\rightarrow \bar{\mathbf{w}} \text{ strongly in } C^\alpha([a + \delta, b - \delta] \times [0, T]), \quad \forall \delta \in (0, (b - a)/2), \quad \alpha \in (0, 1/4) \\ \mathbf{w}_t &\rightharpoonup \bar{\mathbf{w}}_t \text{ weakly in } L^2(Q_T), \\ \mathbf{w}_x &\rightharpoonup \bar{\mathbf{w}}_x \text{ weakly} - * \text{ in } L^\infty(0, T; L^2(a + \delta, b - \delta)), \quad \forall \delta \in (0, (b - a)/2), \\ \mathbf{w} &\rightarrow \bar{\mathbf{w}} \text{ strongly in } L^r(Q_T), \quad \forall r \in [1, +\infty), \\ \sqrt{\mu} \|\mathbf{w}_x\|_{L^4(Q_T)} &\rightarrow 0. \end{aligned}$$

Moreover, $(\bar{\rho}, \bar{u}, \bar{\mathbf{w}}, \bar{\mathbf{b}}, \bar{\theta})$ solves problem (1.1)–(1.3) with $\mu = 0$ in the sense:

$$\left. \begin{aligned} \bar{\rho}_t + (\bar{\rho} \bar{u})_x &= 0, \\ (\bar{\rho} \bar{u})_t + (\bar{\rho} \bar{u}^2 + \gamma \bar{\rho} \bar{\theta} + |\bar{\mathbf{b}}|^2/2)_x &= \lambda \bar{u}_{xx}, \\ (\bar{\rho} \bar{\mathbf{w}})_t + (\bar{\rho} \bar{u} \bar{\mathbf{w}} - \bar{\mathbf{b}})_x &= 0, \\ \bar{\mathbf{b}}_t + (\bar{u} \bar{\mathbf{b}} - \bar{\mathbf{w}})_x &= \nu \bar{\mathbf{b}}_{xx}, \\ (\bar{\rho} \bar{\theta})_t + (\bar{\rho} \bar{u} \bar{\theta})_x + \gamma \bar{\rho} \bar{\theta} \bar{u}_x - [\kappa(\bar{\rho}, \bar{\theta}) \bar{\theta}_x]_x &= \lambda \bar{u}_x^2 + \nu |\bar{\mathbf{b}}_x|^2, \end{aligned} \right\} \text{ a.e. in } Q_T, \quad (1.10)$$

$$\iint_{Q_T} \left\{ [(\bar{\rho} \bar{\theta})_t + (\bar{\rho} \bar{u} \bar{\theta})_x + \gamma \bar{\rho} \bar{\theta} \bar{u}_x - \lambda \bar{u}_x^2 - \nu |\bar{\mathbf{b}}_x|^2] \varphi + \kappa(\bar{\rho}, \bar{\theta}) \bar{\theta}_x \varphi_x \right\} dx dt = 0,$$

for all $\varphi \in L^2(0, T; W^{1,2}(\Omega))$.

(iii) Assume that $(\bar{\rho}, \bar{u}, \bar{\mathbf{w}}, \bar{\mathbf{b}}, \bar{\theta}) \in \mathbb{F}$ is a solution for the limit problem (1.10). Then

$$\begin{aligned} &\|(\rho - \bar{\rho}, u - \bar{u}, \mathbf{w} - \bar{\mathbf{w}}, \mathbf{b} - \bar{\mathbf{b}}, \theta - \bar{\theta})\|_{L^\infty(0, T; L^2(\Omega))} \\ &\quad + \|(u_x - \bar{u}_x, \mathbf{b}_x - \bar{\mathbf{b}}_x, \theta_x - \bar{\theta}_x)\|_{L^2(Q_T)} = O(\mu^{1/4}). \end{aligned}$$

Remark 1.1. With the estimates appearing in the above theorem, and following the argument given in [20](cf. [3]), if the initial data is in Hölder space, i.e.,

$$\rho_0 \in C^{1+\alpha}(\Omega), \quad (u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0) \in C^{2+\alpha}(\Omega)$$

for some $\alpha \in (0, 1)$, then there exists a unique classical solution

$$\rho \in C^{1+\alpha, 1+\alpha/2}(Q_T), \quad (u, \mathbf{w}, \mathbf{b}, \theta) \in C^{2+\alpha, 1+\alpha/2}(Q_T),$$

and it satisfies (1.9).

Remark 1.2. It should be pointed out that if we only consider the global existence with fixed μ , then the condition $u_0 \in W^{2,m}(m > 1)$ in (1.8) can be removed. In fact, this can be done in a more easy way, but we do not pursue it in the paper.

Compared to [7,9] and some related references, the generality of the condition (1.4) causes some other technical difficulties since all the estimates in (1.9) must be uniform in μ . Firstly, we must overcome the difficulty coming from the dissipative estimate on the temperature. For example, Fan-Jiang-Nakamura [7] only established the μ -uniform estimate of θ_x in $L^\beta(Q_T)$ with any $\beta \in (1, 3/2)$ by means of the technique used by Frid and Shelukhin [11]. Secondly, to obtain the stronger convergence of \mathbf{w} and \mathbf{b} (see Theorem 1.1(ii)), we must establish some new uniform estimates on the derivatives of \mathbf{w} and \mathbf{b} . Thirdly, we must seek a new method to obtain a uniform upper bound of the temperature.

To overcome the difficulties, some techniques are developed here. One of two ingredients in the proof is the boundary estimates of derivatives of the transverse velocity and the magnetic field, and the other is that we deduce a uniform upper bound of θ by a simple, direct method.

Below we present a sketch of the proof to (1.9). Firstly, the uniform upper and lower bounds of the density can be obtained in a standard way. Next, a key observation is that we can establish the uniform bound of $\|u_{xx}\|_{L^{m_0}(Q_T)}$ ($m_0 > 1$) by L^p -theory of linear parabolic equations (see Lemma 2.5), which plays an important role in this paper. It should be pointed out that it is in this step we ask the condition $u_0 \in W^{2,m}(\Omega)$ for some $m > 1$. By virtue of the estimate and a delicate analysis, we then deduce the difficult bounds of $\|\omega \mathbf{w}_x\|_{L^\infty(0,T;L^2(\Omega))}$ and $\|(u_t, \mathbf{b}_t, \mathbf{w}_t, u_{xx}, \theta_x, \omega \mathbf{b}_{xx})\|_{L^2(Q_T)}$ (see Lemma 2.10). In this step, the main idea is to use the norm $\|u_{xx}\|_{L^2(Q_T)}$ to control the qualities $\|(\omega \mathbf{w}_x, \mathbf{b}_x)\|_{L^\infty(0,T;L^2(\Omega))}$ and $\|\mathbf{w}_t\|_{L^2(Q_T)}$ (see Lemmas 2.7-2.9) and then, from the equations of u and θ it follows the uniform bound of $\|u_{xx}\|_{L^2(Q_T)}$ by Gronwall's inequality. With the uniform bound of $\|\mathbf{b}_t\|_{L^2(Q_T)}$, we deduce the uniform bounds of $\|\mathbf{w}_x\|_{L^\infty(0,T;L^1(\Omega))}$ and $\|\mathbf{b}_x\|_{L^2(0,T;L^\infty(\Omega))}$ (see Lemmas 2.11 and 2.12), by which we further obtain the uniform bounds of $\|\sqrt{\omega} \mathbf{w}_x\|_{L^\infty(0,T;L^2(\Omega))}$ and $(\mu^{1/4} \|\mathbf{w}_x\|_{L^\infty(0,T;L^2(\Omega))} + \mu^{3/4} \|\mathbf{w}_{xx}\|_{L^2(Q_T)})$ (see Lemma 2.13), which are essential to study both L^2 convergence rate and boundary layer thickness. Due to the above estimates, we finally get an upper bound of θ in a direct way (see Lemma 2.14). As a consequence, the uniform bound of $\|(\theta_t, \theta_{xx})\|_{L^2(Q_T)}$ can be obtained by a brief argument (see Lemma 2.15). Consequently, the passage to limit is justified in the more strong sense.

Next, we investigate the thickness of boundary layer. At first, we give the definition of a BL-thickness defined as in [12] (cf. [17]).

Definition 1.2. A function $\delta(\mu)$ is called a BL-thickness for problem (1.1)-(1.3) with vanishing μ if $\delta(\mu) \downarrow 0$ as $\mu \downarrow 0$, and

$$\begin{aligned} \lim_{\mu \rightarrow 0} \|(\rho - \bar{\rho}, u - \bar{u}, \mathbf{w} - \bar{\mathbf{w}}, \mathbf{b} - \bar{\mathbf{b}}, \theta - \bar{\theta})\|_{L^\infty(0,T;L^\infty(\Omega_{\delta(\mu)}))} &= 0, \\ \inf_{\mu \rightarrow 0} \|(\rho - \bar{\rho}, u - \bar{u}, \mathbf{w} - \bar{\mathbf{w}}, \mathbf{b} - \bar{\mathbf{b}}, \theta - \bar{\theta})\|_{L^\infty(0,T;L^\infty(\Omega))} &> 0, \end{aligned}$$

where $\Omega_\delta = (\delta, 1 - \delta)$ for $\delta \in (0, 1/2)$, and $(\rho, u, \mathbf{w}, \mathbf{b}, \theta)$ and $(\bar{\rho}, \bar{u}, \bar{\mathbf{w}}, \bar{\mathbf{b}}, \bar{\theta})$ are the solutions to problem (1.1)-(1.3) and problem (1.1)-(1.3) with $\mu = 0$, respectively.

We shall prove that for any $\alpha \in (0, 1/2)$, the function $\delta(\mu) = \mu^\alpha$ is a BL-thickness, which is almost optimal since it is close to the classical value $O(\sqrt{\mu})$ (see e.g. [26]). One can see from the proof in Section 3 that our method is a bit different from that used in [12,17], which is based on an iteration inequality (3.8). To indicate the idea clearly, we further assume that

$$\mathbf{w}_0 = \mathbf{b}_0 \equiv \mathbf{0}. \quad (1.11)$$

Theorem 1.3. *Let (1.4), (1.8) and (1.11) hold. Assume that $(\mathbf{w}^-, \mathbf{w}^+)$ is not identically equal to $\mathbf{0}$. Then the limit problem (1.10) has a unique solution $(\bar{\rho}, \bar{u}, \mathbf{0}, \mathbf{0}, \bar{\theta})$ in \mathbb{F} , and the function $\delta(\mu) = \mu^\alpha$ for any $\alpha \in (0, 1/2)$ is a BL-thickness for problem (1.1)-(1.3) such that*

$$\begin{aligned} \lim_{\mu \rightarrow 0} \|(\rho - \bar{\rho}, u - \bar{u}, \mathbf{b}, \theta - \bar{\theta})\|_{C^\alpha(\bar{Q}_T)} &= 0, \quad \forall \alpha \in (0, 1/4), \\ \lim_{\mu \rightarrow 0} \|\mathbf{w}\|_{L^\infty(0,T;L^\infty(\delta(\mu), 1-\delta(\mu)))} &= 0, \quad \inf_{\mu \rightarrow 0} \lim_{\mu \rightarrow 0} \|\mathbf{w}\|_{L^\infty(0,T;L^\infty(\Omega))} > 0. \end{aligned}$$

Moreover, \mathbf{w} has the asymptotic property:

$$\|\mathbf{w}_x\|_{L^\infty(0,T;L^2(\delta, 1-\delta))}^2 \leq \begin{cases} C_n(\tau + \tau^3 + \dots + \tau^{n-2}) + C_n\mu^{(n-1)/2}/\delta^n & (n = \text{odd}), \\ C_n(\tau + \tau^3 + \dots + \tau^{n-1}) + C_n\mu^{(n-1)/2}/\delta^n & (n = \text{even}), \end{cases}$$

where $\delta \in (0, 1/2)$, $\tau = \sqrt{\mu}/\delta$, and the constants C_n are independent of μ and δ .

The remainder of this paper shall be arranged as follows. In Section 2, we will prove Theorem 1.1. For this, a lot of a priori estimates independent of μ are derived in Section 2.1, which are sufficient to prove this theorem. The second and third parts of this theorem can be shown in Sections 2.2 and 2.3, respectively. Finally, we will give the proof of Theorem 1.3 in Section 3.

2 The proof of Theorem 1.1

The existence and uniqueness of local solutions can be obtained by using the Banach theorem and the contractivity of the operator defined by the linearization of the problem on a small time interval (cf. [24, 30]). The existence of global solutions is proved by extending the local solutions globally in time based on the global a priori estimates of solutions. The uniqueness of the global solution follows from the uniqueness of the local solution. Thus, the next subsection will focus on deriving required a priori estimates of the solution $(\rho, u, \mathbf{w}, \mathbf{b}, \theta)$. Moreover, all a priori estimates which will be established are uniform in μ .

Throughout this section, we shall denote by C the various positive constants dependent on T , but independent of μ .

2.1 A priori estimates independent of μ

Rewrite (1.1) as

$$\begin{aligned} \mathcal{E}_t + \left[u(\mathcal{E} + p + \frac{1}{2}|\mathbf{b}|^2) - \mathbf{w} \cdot \mathbf{b} \right]_x &= (\lambda u u_x + \mu \mathbf{w} \cdot \mathbf{w}_x + \nu \mathbf{b} \cdot \mathbf{b}_x + \kappa \theta_x)_x, \\ (\rho \mathcal{S})_t + (\rho u \mathcal{S})_x - \left(\frac{\kappa \theta_x}{\theta} \right)_x &= \frac{\lambda u_x^2 + \mu |\mathbf{w}_x|^2 + \nu |\mathbf{b}_x|^2}{\theta} + \frac{\kappa \theta_x^2}{\theta^2}, \end{aligned} \tag{2.1}$$

where \mathcal{E} and \mathcal{S} are the total energy and the entropy, respectively,

$$\mathcal{E} = \rho \left[\theta + \frac{1}{2}(u^2 + |\mathbf{w}|^2) \right] + \frac{1}{2}|\mathbf{b}|^2, \quad \mathcal{S} = \ln \theta - \gamma \ln \rho.$$

Lemma 2.1. *Let (1.4) and (1.8) hold. Then*

$$\begin{aligned} \int_{\Omega} \rho(x, t) dx &= \int_{\Omega} \rho_0(x) dx, \quad \forall t \in (0, T), \\ \sup_{0 < t < T} \int_{\Omega} [\rho(\theta + u^2 + |\mathbf{w}|^2) + |\mathbf{b}|^2] dx &\leq C, \\ \iint_{Q_T} \left(\frac{\lambda u_x^2 + \mu |\mathbf{w}_x|^2 + \nu |\mathbf{b}_x|^2}{\theta} + \frac{\kappa \theta_x^2}{\theta^2} \right) dx dt &\leq C. \end{aligned} \quad (2.2)$$

Proof. Integrating (2.1)₁ over $Q_t = \Omega \times (0, t)$ with $t \in (0, T)$ yields

$$\int_{\Omega} \mathcal{E} dx = \int_{\Omega} \mathcal{E}|_{t=0} dx + \mu \int_0^t \mathbf{w} \cdot \mathbf{w}_x|_{x=0}^{x=1} ds. \quad (2.3)$$

Let $a = 0$ or 1 . We first integrate (1.1)₃ from $x = a$ to x , and then integrate the resulting equation over Ω , to obtain

$$\mu \mathbf{w}_x(a, t) = \mu (\mathbf{w}^+ - \mathbf{w}^-) - \int_{\Omega} (\rho u \mathbf{w} - \mathbf{b}) dx - \frac{\partial}{\partial t} \int_{\Omega} \int_a^x \rho \mathbf{w} dy dx.$$

Multiplying it by $\mathbf{w}(a, t)$ and integrating over $(0, t)$, we have

$$\begin{aligned} \mu \int_0^t (\mathbf{w} \cdot \mathbf{w}_x)(a, s) ds &= \mu \int_0^t (\mathbf{w}^+ - \mathbf{w}^-) \cdot \mathbf{w}(a, s) ds - \int_0^t \mathbf{w}(a, s) \cdot \left(\int_{\Omega} (\rho u \mathbf{w} - \mathbf{b}) dx \right) ds \\ &\quad - \mathbf{w}(a, t) \cdot \left(\int_{\Omega} \int_a^x \rho \mathbf{w} dy dx \right) + \mathbf{w}(a, 0) \cdot \left(\int_{\Omega} \int_a^x \rho_0 \mathbf{w}_0 dy dx \right) \\ &\quad + \int_0^t \mathbf{w}_t(a, t) \cdot \left(\int_{\Omega} \int_a^x \rho \mathbf{w} dy dx \right) dt, \end{aligned}$$

hence, by Young's inequality and (2.2)₁,

$$\begin{aligned} \left| \mu \int_0^t (\mathbf{w} \cdot \mathbf{w}_x)(a, s) ds \right| &\leq C + C \int_{\Omega} \rho |\mathbf{w}| dx + C \iint_{Q_t} (\rho |u| |\mathbf{w}| + |\mathbf{b}| + \rho |\mathbf{w}|) dx ds \\ &\leq C + \frac{1}{2} \int_{\Omega} \mathcal{E} dx + C \iint_{Q_t} \mathcal{E} dx ds. \end{aligned}$$

Substituting it into (2.3) yields

$$\int_{\Omega} \mathcal{E} dx \leq C + C \iint_{Q_t} \mathcal{E} dx ds,$$

and so, (2.2)₂ follows from Gronwall's inequality.

(2.2)₃ can be proved by integrating (2.1)₂ and using (2.2)₂. The proof is complete. \square

From Lemma 2.1, the following estimates can be proved.

Lemma 2.2. *Let (1.4) and (1.8) hold. Then*

$$\begin{aligned}
C^{-1} &\leq \rho \leq C, \\
\theta &\geq C, \\
\iint_{Q_T} \frac{\kappa \theta_x^2}{\theta^{1+\alpha}} dx dt &\leq C, \quad \forall \alpha \in (0, \min\{1, q\}), \\
\int_0^T \|\theta\|_{L^\infty(\Omega)}^{q+1-\alpha} dt &\leq C, \quad \forall \alpha \in (0, \min\{1, q\}), \\
\iint_{Q_T} (\lambda u_x^2 + \mu |\mathbf{w}_x|^2 + \nu |\mathbf{b}_x|^2) dx dt &\leq C, \\
\int_0^T \|\mathbf{b}\|_{L^\infty(\Omega)}^2 dt &\leq C, \\
\iint_{Q_T} |\theta_x|^{3/2} dx dt &\leq C.
\end{aligned} \tag{2.4}$$

Proof. The proofs to the estimates $\rho \leq C$ and (2.4)₃-(2.4)₅ can be found in [9] where the vacuum is permitted. (2.4)₆ is an immediate consequence of (2.4)₅, so the estimate $\rho \geq C^{-1}$ can be proved in a standard way (see [7]). We omit their proofs for brevity.

Now we turn to (2.4)₂, whose proof depends only on the estimate $\rho \leq C$. It follows from (1.1)₅ that

$$\theta_t + u\theta_x - \frac{1}{\rho}(\kappa\theta_x)_x \geq \frac{\lambda}{\rho} \left(u_x^2 - \frac{p}{\lambda} u_x \right) = \frac{\lambda}{\rho} \left(u_x - \frac{p}{2\lambda} \right)^2 - \frac{\gamma^2}{4\lambda} \rho \theta^2.$$

By $\rho \leq C$, we have

$$\theta_t + u\theta_x - \frac{1}{\rho}(\kappa\theta_x)_x + K\theta^2 \geq 0,$$

where K is a positive constant independent of μ . Let $z = \theta - \underline{\theta}$, where $\underline{\theta} = \frac{\min_{\overline{\Omega}} \theta_0}{Ct+1}$ with $C = K \min_{\overline{\Omega}} \theta_0$. Then $z_x|_{x=0,1} = 0$, $z|_{t=0} \geq 0$, and

$$\begin{aligned}
&z_t + uz_x - \frac{1}{\rho}(\kappa z_x)_x + K(\theta + \underline{\theta})z \\
&= \theta_t + C \frac{\min_{\overline{\Omega}} \theta_0}{(Ct+1)^2} + u\theta_x - \frac{1}{\rho}(\kappa\theta_x)_x + K\theta^2 - K \left(\frac{\min_{\overline{\Omega}} \theta_0}{Ct+1} \right)^2 \\
&\geq C \frac{\min_{\overline{\Omega}} \theta_0}{(Ct+1)^2} - K \left(\frac{\min_{\overline{\Omega}} \theta_0}{Ct+1} \right)^2 = 0,
\end{aligned}$$

and then, $z \geq 0$ on \overline{Q}_T by the comparison theorem, so (2.4)₂.

It remains to show (2.4)₇. By (2.4)₂ and (2.4)₃, we have

$$\iint_{Q_T} \frac{\theta_x^2}{\theta} dx dt \leq C. \tag{2.5}$$

Then, we have by the mean value theorem, Lemma 2.1, (2.4)₁ and Hölder's inequality

$$\begin{aligned}
\theta &\leq \int_{\Omega} \theta dx + \int_{\Omega} |\theta_x| dx \\
&\leq C + C \left(\int_{\Omega} \frac{\theta_x^2}{\theta} dx \right)^{1/2} \left(\int_{\Omega} \theta dx \right)^{1/2} \\
&\leq C + C \left(\int_{\Omega} \frac{\theta_x^2}{\theta} dx \right)^{1/2},
\end{aligned}$$

which together with (2.5) gives

$$\int_0^T \|\theta\|_{L^\infty(\Omega)}^2 dt \leq C. \quad (2.6)$$

Thus, it follows from Hölder's inequality, Lemma 2.1 and (2.6) that

$$\begin{aligned}
\iint_{Q_T} |\theta_x|^{3/2} dx dt &\leq \left(\iint_{Q_T} \frac{\theta_x^2}{\theta} dx dt \right)^{3/4} \left(\iint_{Q_T} \theta^3 dx dt \right)^{1/4} \\
&\leq C \left(\int_0^T \|\theta^2\|_{L^\infty(\Omega)} \int_{\Omega} \theta dx dt \right)^{1/4} \leq C.
\end{aligned}$$

The proof is complete. □

About the magnetic field \mathbf{b} , we have

Lemma 2.3. *Let (1.4) and (1.8) hold. Then*

$$\sup_{0 < t < T} \int_{\Omega} |\mathbf{b}|^4 dx + \iint_{Q_T} |\mathbf{b}|^2 |\mathbf{b}_x|^2 dx dt \leq C.$$

Proof. Multiplying (1.1)₄ by $4|\mathbf{b}|^2 \mathbf{b}$ and integrating over Q_t , we obtain

$$\begin{aligned}
&\int_{\Omega} |\mathbf{b}|^4 dx + 4\nu \iint_{Q_t} |\mathbf{b}|^2 |\mathbf{b}_x|^2 dx ds + 8\nu \iint_{Q_t} |\mathbf{b} \cdot \mathbf{b}_x|^2 dx ds \\
&= \int_{\Omega} |\mathbf{b}_0|^4 dx + 4 \iint_{Q_t} \mathbf{w}_x \cdot (|\mathbf{b}|^2 \mathbf{b}) dx ds - 4 \iint_{Q_t} (u\mathbf{b})_x \cdot (|\mathbf{b}|^2 \mathbf{b}) dx ds.
\end{aligned} \quad (2.7)$$

Integrating by parts and using Young's inequality, we have

$$\begin{aligned}
\iint_{Q_t} \mathbf{w}_x \cdot (|\mathbf{b}|^2 \mathbf{b}) dx ds &= - \iint_{Q_t} \mathbf{w} \cdot (\mathbf{b}_x |\mathbf{b}|^2) dx ds - 2 \iint_{Q_t} (\mathbf{w} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{b}_x) dx ds \\
&\leq 3 \iint_{Q_t} |\mathbf{w}| |\mathbf{b}|^2 |\mathbf{b}_x| dx ds \\
&\leq \frac{\nu}{4} \iint_{Q_t} |\mathbf{b}|^2 |\mathbf{b}_x|^2 dx ds + C \iint_{Q_t} |\mathbf{w}|^2 |\mathbf{b}|^2 dx ds \\
&\leq \frac{\nu}{4} \iint_{Q_t} |\mathbf{b}|^2 |\mathbf{b}_x|^2 dx ds + C \int_0^t \|\mathbf{b}\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\mathbf{w}|^2 dx ds \\
&\leq \frac{\nu}{4} \iint_{Q_t} |\mathbf{b}|^2 |\mathbf{b}_x|^2 dx ds + C,
\end{aligned} \quad (2.8)$$

where we used (2.2)₂ and (2.4)₆.

On the other hand, we have

$$\begin{aligned}
& - \iint_{Q_t} (u\mathbf{b})_x \cdot |\mathbf{b}|^2 \mathbf{b} dx ds = 3 \iint_{Q_t} u(\mathbf{b}_x \cdot \mathbf{b}) |\mathbf{b}|^2 dx ds \\
& \leq \frac{\nu}{4} \iint_{Q_t} |\mathbf{b}|^2 |\mathbf{b}_x|^2 dx ds + C \iint_{Q_t} u^2 |\mathbf{b}|^4 dx ds \\
& \leq \frac{\nu}{4} \iint_{Q_t} |\mathbf{b}|^2 |\mathbf{b}_x|^2 dx ds + C \int_0^t \|u^2\|_{L^\infty(\Omega)} \int_\Omega |\mathbf{b}|^4 dx ds.
\end{aligned} \tag{2.9}$$

Plugging (2.8) and (2.9) into (2.7) and using Gronwall's inequality, we finish the proof by noticing $\int_0^T \|u^2\|_{L^\infty(\Omega)} dt \leq C \iint_{Q_T} u_x^2 dx dt \leq C$. \square

Lemma 2.4. *Let (1.4) and (1.8) hold. Then*

$$\begin{aligned}
& \sup_{0 < t < T} \int_\Omega \rho_x^2 dx + \iint_{Q_T} (\rho_t^2 + \theta \rho_x^2) dx dt \leq C, \\
& |\rho(x, t) - \rho(y, s)| \leq C (|x - y|^{1/2} + |s - t|^{1/4}), \quad \forall (x, t), (y, s) \in \overline{Q_T}.
\end{aligned} \tag{2.10}$$

Proof. Set $\eta = 1/\rho$. It follows from the equation (1.1)₁ that

$$u_x = \rho(\eta_t + u\eta_x).$$

Substituting it into (1.1)₂ yields

$$[\rho(u - \lambda\eta_x)]_t + [\rho u(u - \lambda\eta_x)]_x = \gamma \rho^2(\theta\eta_x - \eta\theta_x) - \mathbf{b} \cdot \mathbf{b}_x.$$

Multiplying it by $(u - \lambda\eta_x)$ and integrating over Q_t , we have

$$\begin{aligned}
& \frac{1}{2} \int_\Omega \rho(u - \lambda\eta_x)^2 dx + \gamma \lambda \iint_{Q_t} \theta \rho^2 \eta_x^2 dx ds \\
& = \frac{1}{2} \int_\Omega \rho_0(u_0 + \lambda \rho_0^{-2} \rho_{0x})^2 dx + \gamma \iint_{Q_t} \rho^2 \theta u \eta_x dx ds \\
& \quad - \gamma \iint_{Q_t} \rho^2 \eta \theta_x (u - \lambda\eta_x) dx ds - \iint_{Q_t} \mathbf{b} \cdot \mathbf{b}_x (u - \lambda\eta_x) dx ds.
\end{aligned}$$

To estimate the second integral on right-hand side, we use Young's inequality, Lemmas 2.1 and 2.2 to obtain

$$\begin{aligned}
\gamma \iint_{Q_t} \rho^2 \theta u \eta_x dx ds & \leq \frac{\gamma \lambda}{2} \iint_{Q_t} \theta \rho^2 \eta_x^2 dx ds + C \iint_{Q_t} \theta u^2 dx ds \\
& \leq \frac{\gamma \lambda}{2} \iint_{Q_t} \theta \rho^2 \eta_x^2 dx ds + C \int_0^t \|\theta\|_{L^\infty(\Omega)} \int_\Omega u^2 dx ds \\
& \leq C + \frac{\gamma \lambda}{2} \iint_{Q_t} \theta \rho^2 \eta_x^2 dx ds.
\end{aligned}$$

On the other hand, we have by Cauchy-Schwarz's inequality, (2.5) and Lemma 2.3

$$\begin{aligned}
& -\gamma \iint_{Q_t} \rho^2 \eta \theta_x (u - \lambda \eta_x) dx ds - \iint_{Q_t} \mathbf{b} \cdot \mathbf{b}_x (u - \lambda \eta_x) dx ds \\
& \leq C + C \iint_{Q_t} \theta \rho (u - \lambda \eta_x)^2 dx ds + C \iint_{Q_t} \frac{\theta_x^2}{\theta} dx ds + C \iint_{Q_t} \rho (u - \lambda \eta_x)^2 dx ds \\
& \leq C + C \int_0^t (1 + \|\theta\|_{L^\infty(\Omega)}) \int_\Omega \rho (u - \lambda \eta_x)^2 dx ds.
\end{aligned}$$

Combining the above results yields

$$\begin{aligned}
& \int_\Omega \rho (u - \lambda \eta_x)^2 dx + \iint_{Q_t} \theta \rho^2 \eta_x^2 dx ds \\
& \leq C + C \int_0^t (1 + \|\theta\|_{L^\infty(\Omega)}) \int_\Omega \rho (u - \lambda \eta_x)^2 dx ds,
\end{aligned}$$

which together with Gronwall's inequality gives

$$\sup_{0 < t < T} \int_\Omega \rho_x^2 dx + \iint_{Q_T} \theta \rho_x^2 dx ds \leq C.$$

By this estimate and Lemma 2.2, we derive from the equation (1.1)₁ that

$$\iint_{Q_T} \rho_t^2 dx dt \leq C \int_0^T \|u^2\|_{L^\infty(\Omega)} \int_\Omega \rho_x^2 dx dt + C \iint_{Q_T} u_x^2 dx dt \leq C.$$

Thus (2.10)₁ holds.

Now we prove the second estimate. Let $\beta(x) = \rho(x, t) - \rho(x, s)$ for any $x \in [0, 1]$ and $s, t \in [0, T]$ with $s \neq t$. Then for any $x \in [0, 1]$ and $\delta \in (0, 1/2]$, there exist some $y \in [0, 1]$ and ξ between x and y such that $\delta = |y - x|$ and $\beta(\xi) = \frac{1}{x-y} \int_y^x \beta(z) dz$, and hence

$$\beta(x) = \frac{1}{x-y} \int_y^x \beta(z) dz + \int_\xi^x \beta'(z) dz,$$

therefore, by Hölder's inequality and (2.1),

$$\begin{aligned}
|\beta(x)| & \leq \frac{1}{\delta} \left| \int_y^x \beta(z) dz \right| + \left| \int_\xi^x \beta'(z) dz \right| \\
& \leq \frac{1}{\delta} \left| \int_y^x \int_s^t \rho_\tau d\tau dz \right| + \left| \int_\xi^x [\rho_z(z, t) - \rho_z(z, s)] dz \right| \\
& \leq \frac{1}{\delta} \left(\iint_{Q_T} \rho_\tau^2 d\tau dz \right)^{1/2} |x - y|^{1/2} |s - t|^{1/2} \\
& \quad + \left(\int_0^1 (|\rho_z(z, s)|^2 + |\rho_z(z, t)|^2) dz \right)^{1/2} |x - \xi|^{1/2} \\
& \leq C \delta^{-1/2} |s - t|^{1/2} + C \delta^{1/2}.
\end{aligned}$$

If $0 < |s - t|^{1/2} < 1/2$, taking $\delta = |s - t|^{1/2}$ yields

$$|\rho(x, s) - \rho(x, t)| \leq C|s - t|^{1/4}. \quad (2.11)$$

If $|s - t|^{1/2} \geq 1/2$, then (2.11) holds since ρ is uniformly bounded in μ .

On the other hand, we have by (2.10)₁

$$|\rho(x, t) - \rho(y, t)| = \left| \int_y^x \rho_z dz \right| \leq C|x - y|^{1/2}. \quad (2.12)$$

Thus, (2.10)₂ is a consequence of the triangle inequality. The proof is complete. \square

To deduce other required μ -uniform estimates, we need the following lemma which plays an important role in this paper.

Lemma 2.5. *Let (1.4) and (1.8) hold. Then*

$$\iint_{Q_T} |u_{xx}|^{m_0} dx dt \leq C, \quad m_0 = \min\{m, 4/3\}. \quad (2.13)$$

In particular,

$$\int_0^T \|u_x\|_{L^\infty(\Omega)}^{m_0} dt \leq C. \quad (2.14)$$

Proof. Note that the estimate (2.14) is an immediate consequence of (2.13). Thus, it is enough to prove (2.13). To this end, we rewrite the equation (1.1)₂ as

$$u_t - \frac{\lambda}{\rho} u_{xx} = -uu_x - \gamma\theta_x - \frac{\gamma}{\rho}\rho_x\theta - \frac{1}{\rho}\mathbf{b} \cdot \mathbf{b}_x =: f. \quad (2.15)$$

We will apply L^p estimates of linear parabolic equations (cf. [22, Theorem 7.17]) to show (2.13). From (2.10)₂, the coefficient $a(x, t) := \lambda/\rho$ satisfies

$$|a(x, t) - a(y, s)| \leq C(|x - y|^{1/2} + |s - t|^{1/4}), \quad \forall (x, t), (y, s) \in \overline{Q_T}.$$

Due to the condition on u_0 in (1.8), we only need to give a uniform bound of f in $L^{4/3}(Q_T)$.

From Lemmas 2.2 and 2.3, the second term and the forth term on right-hand side of (2.15) are uniformly bounded in $L^{3/2}(Q_T)$ and $L^2(Q_T)$, respectively.

To deal with the first term on right-hand side of (2.15), we observe by Hölder inequality and Lemma 2.1

$$u^2 \leq 2 \int_{\Omega} |uu_x| dx \leq 2 \left(\int_{\Omega} u^2 dx \right)^{1/2} \left(\int_{\Omega} u_x^2 dx \right)^{1/2} \leq C \left(\int_{\Omega} u_x^2 dx \right)^{1/2},$$

therefore, we have by Lemma 2.2

$$\int_0^T \|u\|_{L^\infty}^4 dt \leq C \iint_{Q_T} u_x^2 dx \leq C,$$

which together with Young's inequality yields

$$\begin{aligned} \iint_{Q_T} |uu_x|^{3/2} dx dt &\leq C \iint_{Q_T} u_x^2 dx dt + C \iint_{Q_T} u^6 dx dt \\ &\leq C + C \int_0^T \|u\|_{L^\infty(\Omega)}^4 \int_\Omega u^2 dx dt \leq C. \end{aligned}$$

As to the third term on right-hand side of (2.15), we have by (2.10)₁ and (2.6)

$$\iint_{Q_T} |\rho_x \theta|^{4/3} dx dt \leq C \iint_{Q_T} \rho_x^2 \theta dx dt + C \iint_{Q_T} \theta^2 dx dt \leq C.$$

Combining the above results gives $\|f\|_{L^{4/3}(Q_T)} \leq C$. The proof is then completed. \square

By a direct application of the above lemma, we obtain

Lemma 2.6. *Let (1.4) and (1.8) hold. Then*

$$\mu \sup_{0 < t < T} \int_\Omega |\mathbf{w}_x|^2 dx + \mu^2 \iint_{Q_T} |\mathbf{w}_{xx}|^2 dx ds \leq C.$$

Proof. We rewrite (1.1)₃ in the form

$$\mathbf{w}_t - \frac{\mu}{\rho} \mathbf{w}_{xx} = \frac{1}{\rho} \mathbf{b}_x - u \mathbf{w}_x, \quad (2.16)$$

and multiply it by $\mu \mathbf{w}_{xx}$ and integrating over Q_t to obtain

$$\begin{aligned} &\frac{\mu}{2} \int_\Omega |\mathbf{w}_x|^2 dx + \mu^2 \iint_{Q_t} \frac{1}{\rho} |\mathbf{w}_{xx}|^2 dx ds \\ &= \frac{\mu}{2} \int_\Omega |\mathbf{w}_{0x}|^2 dx - \mu \iint_{Q_t} \frac{1}{\rho} \mathbf{b}_x \cdot \mathbf{w}_{xx} dx ds \\ &\quad - \frac{\mu}{2} \iint_{Q_t} u_x |\mathbf{w}_x|^2 dx ds + \mu \int_0^t \mathbf{w}_t \cdot \mathbf{w}_x \Big|_{x=0}^{x=1} ds \\ &\leq C\mu + \frac{\mu^2}{4} \iint_{Q_t} \frac{1}{\rho} |\mathbf{w}_{xx}|^2 dx ds + C \iint_{Q_t} |\mathbf{b}_x|^2 dx ds \\ &\quad + C \int_0^t \|u_x\|_{L^\infty(\Omega)} \left(\mu \int_\Omega |\mathbf{w}_x|^2 dx \right) ds + C\mu \int_0^t \|\mathbf{w}_x\|_{L^\infty(\Omega)} ds. \end{aligned} \quad (2.17)$$

From the mean value theorem and Hölder inequality, we obtain

$$\begin{aligned} |\mathbf{w}_x|^2 &\leq \left| \frac{\mathbf{w}(b, t) - \mathbf{w}(a, t)}{b - a} \right|^2 + 2 \int_\Omega |\mathbf{w}_x| |\mathbf{w}_{xx}| dx \\ &\leq C + C \left(\int_\Omega |\mathbf{w}_x|^2 dx \right)^{1/2} \left(\int_\Omega |\mathbf{w}_{xx}|^2 dx \right)^{1/2}, \end{aligned} \quad (2.18)$$

and so, Young's inequality yields

$$\begin{aligned} \mu \int_0^t \|\mathbf{w}_x\|_{L^\infty(\Omega)} ds &\leq C\mu + C \int_0^t \mu^{1/4} \left(\mu \int_\Omega |\mathbf{w}_x|^2 dx \right)^{1/4} \left(\mu^2 \int_\Omega |\mathbf{w}_{xx}|^2 dx \right)^{1/4} ds \\ &\leq C\sqrt{\mu} + \frac{C\mu}{\epsilon} \iint_{Q_t} |\mathbf{w}_x|^2 dx ds + \epsilon \mu^2 \iint_{Q_t} \frac{1}{\rho} |\mathbf{w}_{xx}|^2 dx ds, \forall \epsilon \in (0, 1). \end{aligned}$$

Inserting it into (2.17) and taking a small $\epsilon > 0$, we find that

$$\begin{aligned} & \mu \int_{\Omega} |\mathbf{w}_x|^2 dx + \mu^2 \iint_{Q_t} \frac{1}{\rho} |\mathbf{w}_{xx}|^2 dx ds \\ & \leq C + C \int_0^t (1 + \|u_x\|_{L^\infty}) \left(\mu \int_{Q_t} |\mathbf{w}_x|^2 dx \right) ds. \end{aligned}$$

Thus, the lemma follows from Gronwall's inequality and (2.14). This proof is complete. \square

Our next main task is to show the other estimates appearing in Theorem 1.1. To this end, we need three preliminary lemmas. The first one reads as

Lemma 2.7. *Let (1.4) and (1.8) hold. Then*

$$\int_{\Omega} |\mathbf{b}_x|^2 \omega^2 dx + \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega^2 dx ds \leq C \iint_{Q_t} |\mathbf{w}_x|^2 \omega^2 dx ds + C \left(\iint_{Q_t} u_{xx}^2 dx ds \right)^{1/2},$$

where ω is the same as that in Theorem 1.1.

Proof. Multiplying (1.1)₄ by $\mathbf{b}_{xx}\omega^2(x)$ and integrating over Q_t , we have

$$\begin{aligned} & - \iint_{Q_t} \mathbf{b}_t \cdot \mathbf{b}_{xx} \omega^2 dx dt + \nu \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega^2 dx ds \\ & = \iint_{Q_t} (u\mathbf{b})_x \cdot \mathbf{b}_{xx} \omega^2 dx ds - \iint_{Q_t} \mathbf{w}_x \cdot \mathbf{b}_{xx} \omega^2 dx ds. \end{aligned} \tag{2.19}$$

To estimate the first integral on left-hand side of (2.19), we integrate by parts and use (1.1)₄ to obtain

$$\begin{aligned} \iint_{Q_t} \mathbf{b}_t \cdot \mathbf{b}_{xx} \omega^2 dx dt & = -\frac{1}{2} \int_{\Omega} |\mathbf{b}_x|^2 \omega^2 dx + \frac{1}{2} \int_{\Omega} |\mathbf{b}_{0x}|^2 \omega^2 dx - 2 \iint_{Q_t} \mathbf{b}_t \cdot \mathbf{b}_x \omega \omega' dx dt \\ & = -\frac{1}{2} \int_{\Omega} |\mathbf{b}_x|^2 \omega^2 dx + \frac{1}{2} \int_{\Omega} |\mathbf{b}_{0x}|^2 \omega^2 dx \\ & \quad - 2 \iint_{Q_t} (\nu \mathbf{b}_{xx} + \mathbf{w}_x - u\mathbf{b}_x - u_x \mathbf{b}) \cdot \mathbf{b}_x \omega \omega' dx ds. \end{aligned} \tag{2.20}$$

Below we deal with the third term on right-hand side of (2.20). By Cauchy-Schwarz's inequality and Lemmas 2.2 and 2.3, we have

$$\begin{aligned} & - 2 \iint_{Q_t} (\nu \mathbf{b}_{xx} + \mathbf{w}_x - u_x \mathbf{b}) \cdot \mathbf{b}_x \omega \omega' dx ds \\ & \leq \frac{\nu}{4} \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega^2 dx ds + C \iint_{Q_t} |\mathbf{b}_x|^2 dx ds + C \iint_{Q_t} |\mathbf{w}_x|^2 \omega^2 dx ds \\ & \quad + C \iint_{Q_t} u_x^2 dx ds + C \iint_{Q_t} |\mathbf{b} \cdot \mathbf{b}_x|^2 dx ds \\ & \leq C + \frac{\nu}{4} \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega^2 dx ds + C \iint_{Q_t} |\mathbf{w}_x|^2 \omega^2 dx ds. \end{aligned}$$

Observe that since from the mean value theorem and $u(1, t) = u(0, t) = 0$, we have

$$|u(x, t)| \leq \|u_x\|_{L^\infty(\Omega)} \omega(x), \tag{2.21}$$

so

$$2 \iint_{Q_t} u |\mathbf{b}_x|^2 \omega \omega' dx ds \leq C \int_0^t \|u_x\|_{L^\infty(\Omega)} \int_\Omega |\mathbf{b}_x|^2 \omega^2 dx ds.$$

Substituting them into (2.20) yields

$$\begin{aligned} \iint_{Q_t} \mathbf{b}_t \cdot \mathbf{b}_{xx} \omega^2 dx dt &\leq C - \frac{1}{2} \int_\Omega |\mathbf{b}_x|^2 \omega^2 dx + \frac{\nu}{4} \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega^2 dx ds \\ &\quad + C \int_0^t \|u_x\|_{L^\infty(\Omega)} \int_\Omega |\mathbf{b}_x|^2 \omega^2 dx ds + C \iint_{Q_t} |\mathbf{w}_x|^2 \omega^2 dx ds. \end{aligned} \quad (2.22)$$

As to the two terms on right-hand side of (2.19), we have by Young's inequality

$$\begin{aligned} &\iint_{Q_t} (u\mathbf{b})_x \cdot \mathbf{b}_{xx} \omega^2 dx ds - \iint_{Q_t} \mathbf{w}_x \cdot \mathbf{b}_{xx} \omega^2 dx ds \\ &\leq \frac{\nu}{4} \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega^2 dx ds + C \iint_{Q_t} |(u\mathbf{b})_x|^2 \omega^2 dx ds + C \iint_{Q_t} |\mathbf{w}_x|^2 \omega^2 dx ds. \end{aligned} \quad (2.23)$$

It remains to treat the second term on right-hand side of (2.23). We observe by Lemma 2.2

$$\int_0^t \|u_x\|_{L^\infty(\Omega)}^2 ds \leq C \iint_{Q_t} |u_x u_{xx}| dx ds \leq C \left(\iint_{Q_t} u_{xx}^2 dx ds \right)^{1/2}, \quad (2.24)$$

which together with Lemma 2.1 gives

$$\begin{aligned} \iint_{Q_t} |(u\mathbf{b})_x|^2 \omega^2 dx ds &\leq C \iint_{Q_t} u^2 |\mathbf{b}_x|^2 \omega^2 dx ds + C \iint_{Q_t} u_x^2 |\mathbf{b}|^2 \omega^2 dx ds \\ &\leq C \int_0^t \|u^2\|_{L^\infty(\Omega)} \int_\Omega |\mathbf{b}_x|^2 \omega^2 dx ds + C \left(\iint_{Q_t} u_{xx}^2 dx ds \right)^{1/2}. \end{aligned}$$

Substituting it into (2.23) and then, substituting the resulting inequality and (2.22) into (2.19) and using Gronwall's inequality, we finish the proof. \square

Lemma 2.8. *Let (1.4) and (1.8) hold. Then*

$$\begin{aligned} \int_\Omega |\mathbf{w}_x|^2 \omega^2 dx + \mu \iint_{Q_t} |\mathbf{w}_{xx}|^2 \omega^2 dx ds &\leq C + C \left(\iint_{Q_t} u_{xx}^2 dx ds \right)^{1/2}, \\ \iint_{Q_t} (|\mathbf{w}_t|^2 + u^2 |\mathbf{w}_x|^2) dx dt &\leq C + C \iint_{Q_t} u_{xx}^2 dx ds. \end{aligned} \quad (2.25)$$

Proof. Multiplying (2.16) by $\mathbf{w}_{xx} \omega^2(x)$ and integrating over Q_t , we have

$$\begin{aligned} &-\iint_{Q_t} \mathbf{w}_t \cdot \mathbf{w}_{xx} \omega^2 dx dt + \mu \iint_{Q_t} |\mathbf{w}_{xx}|^2 \frac{\omega^2}{\rho} dx ds \\ &= \iint_{Q_t} u \mathbf{w}_x \cdot \mathbf{w}_{xx} \omega^2 dx ds - \iint_{Q_t} \mathbf{b}_x \cdot \mathbf{w}_{xx} \frac{\omega^2}{\rho} dx ds. \end{aligned} \quad (2.26)$$

Integrating by parts and using (2.16), we have

$$\begin{aligned}
& \iint_{Q_t} \mathbf{w}_t \cdot \mathbf{w}_{xx} \omega^2 dx dt \\
&= -\frac{1}{2} \int_{\Omega} |\mathbf{w}_x|^2 \omega^2 dx + \frac{1}{2} \int_{\Omega} |\mathbf{w}_{0x}|^2 \omega^2 dx - 2 \iint_{Q_t} \mathbf{w}_t \cdot \mathbf{w}_x \omega \omega' dx dt \\
&= -\frac{1}{2} \int_{\Omega} |\mathbf{w}_x|^2 \omega^2 dx + \frac{1}{2} \int_{\Omega} |\mathbf{w}_{0x}|^2 \omega^2 dx \\
&\quad - 2 \iint_{Q_t} \left(\frac{\mu}{\rho} \mathbf{w}_{xx} - u \mathbf{w}_x + \frac{\mathbf{b}_x}{\rho} \right) \cdot \mathbf{w}_x \omega \omega' dx ds \\
&\leq C - \frac{1}{2} \int_{\Omega} |\mathbf{w}_x|^2 \omega^2 dx + C \mu^2 \iint_{Q_t} |\mathbf{w}_{xx}|^2 dx ds + C \iint_{Q_t} |\mathbf{w}_x|^2 \omega^2 dx ds \\
&\quad + C \iint_{Q_t} |\mathbf{b}_x|^2 dx ds + C \iint_{Q_t} |u| |\mathbf{w}_x|^2 \omega dx ds.
\end{aligned}$$

From (2.21) it follows that

$$\iint_{Q_t} |u| |\mathbf{w}_x|^2 \omega dx ds \leq \int_0^t \|u_x\|_{L^\infty(\Omega)} \int_{\Omega} |\mathbf{w}_x|^2 \omega^2 dx ds,$$

which together with Lemmas 2.2 and 2.6 gives

$$\begin{aligned}
& \iint_{Q_t} \mathbf{w}_t \cdot \mathbf{w}_{xx} \omega^2 dx dt \\
&\leq C - \frac{1}{2} \int_{\Omega} |\mathbf{w}_x|^2 \omega^2 dx + C \int_0^t (1 + \|u_x\|_{L^\infty(\Omega)}) \int_{\Omega} |\mathbf{w}_x|^2 \omega^2 dx ds.
\end{aligned}$$

To estimate the right-hand side of (2.26), we have by integrating by parts and by (2.21)

$$\begin{aligned}
\iint_{Q_t} u \mathbf{w}_x \cdot \mathbf{w}_{xx} \omega^2 dx ds &= -\frac{1}{2} \iint_{Q_t} |\mathbf{w}_x|^2 [u_x \omega^2 + 2u \omega \omega'] dx ds \\
&\leq C \int_0^t \|u_x\|_{L^\infty(\Omega)} \int_{\Omega} |\mathbf{w}_x|^2 \omega^2 dx ds,
\end{aligned}$$

and

$$\begin{aligned}
& - \iint_{Q_t} \mathbf{b}_x \cdot \mathbf{w}_{xx} \frac{\omega^2}{\rho} dx ds \\
&= \iint_{Q_t} \mathbf{w}_x \cdot \mathbf{b}_{xx} \frac{\omega^2}{\rho} dx ds + 2 \iint_{Q_t} \mathbf{w}_x \cdot \mathbf{b}_x \frac{\omega \omega'}{\rho} dx ds - \iint_{Q_t} \mathbf{w}_x \cdot \mathbf{b}_x \frac{\omega^2 \rho_x}{\rho^2} dx ds \\
&\leq C \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega^2 dx ds + C \iint_{Q_t} |\mathbf{w}_x|^2 \omega^2 dx ds + C \iint_{Q_t} |\mathbf{b}_x|^2 dx ds \\
&\quad + C \iint_{Q_t} |\mathbf{b}_x|^2 \omega^2 \rho_x^2 dx ds \\
&\leq C + C \iint_{Q_t} |\mathbf{w}_x|^2 \omega^2 dx ds + C \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega^2 dx ds,
\end{aligned}$$

where we used the fact by Lemmas 2.2 and 2.4

$$\begin{aligned}
\iint_{Q_t} |\mathbf{b}_x|^2 \omega^2 \rho_x^2 dx ds &\leq C \int_0^t \| |\mathbf{b}_x|^2 \omega^2 \|_{L^\infty(\Omega)} ds \\
&\leq C \iint_{Q_t} |(|\mathbf{b}_x|^2 \omega^2)_x| dx ds \\
&\leq C \iint_{Q_t} |\mathbf{b}_x|^2 |\omega \omega'| dx ds + C \iint_{Q_t} |\mathbf{b}_x \cdot \mathbf{b}_{xx}| \omega^2 dx ds \\
&\leq C + C \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega^2 dx ds.
\end{aligned}$$

Substituting the above results into (2.26) and using Lemma 2.7, we have

$$\begin{aligned}
&\int_{\Omega} |\mathbf{w}_x|^2 \omega^2 dx + \mu \iint_{Q_t} |\mathbf{w}_{xx}|^2 \omega^2 dx ds \\
&\leq C + C \int_0^t (1 + \|u_x\|_{L^\infty(\Omega)}) \int_{\Omega} |\mathbf{w}_x|^2 \omega^2 dx ds + C \left(\iint_{Q_t} u_{xx}^2 dx ds \right)^{1/2}.
\end{aligned}$$

Thus, the first estimate of this lemma follows from Gronwall's inequality and (2.14).

Consequently, we have by (2.21), the first estimate of this lemma and (2.24)

$$\begin{aligned}
\iint_{Q_T} u^2 |\mathbf{w}_x|^2 dx dt &\leq \int_0^T \|u_x\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\mathbf{w}_x|^2 \omega^2 dx dt \\
&\leq C \int_0^T \|u_x\|_{L^\infty(\Omega)}^2 dt \left[1 + \left(\iint_{Q_T} u_{xx}^2 dx dt \right)^{1/2} \right] \\
&\leq C + C \iint_{Q_T} u_{xx}^2 dx dt.
\end{aligned}$$

Furthermore, by Lemmas 2.2 and 2.6, we derive from (2.16) that

$$\iint_{Q_T} |\mathbf{w}_t|^2 dx dt \leq C + C \iint_{Q_T} u^2 |\mathbf{w}_x|^2 dx dt \leq C + C \iint_{Q_T} u_{xx}^2 dx dt.$$

The proof is complete. \square

Lemma 2.9. *Let (1.4) and (1.8) hold. Then*

$$\int_{\Omega} |\mathbf{b}_x|^2 dx + \iint_{Q_t} |\mathbf{b}_t|^2 dx dt \leq C + C \left(\iint_{Q_t} u_{xx}^2 dx ds \right)^{1/2}. \quad (2.27)$$

Proof. Multiplying (1.1)₄ by \mathbf{b}_t and integrating over Q_t yield

$$\frac{\nu}{2} \int_{\Omega} |\mathbf{b}_x|^2 dx + \iint_{Q_t} |\mathbf{b}_t|^2 dx dt = \frac{\nu}{2} \int_{\Omega} |\mathbf{b}_{0x}|^2 dx + \iint_{Q_t} [\mathbf{w}_x - (u\mathbf{b})_x] \cdot \mathbf{b}_t dx dt. \quad (2.28)$$

Integrating by parts yields

$$\begin{aligned}
\iint_{Q_t} \mathbf{w}_x \cdot \mathbf{b}_t dx dt &= \int_{\Omega} \mathbf{w}_x \cdot \mathbf{b} dx - \int_{\Omega} \mathbf{w}_{0x} \cdot \mathbf{b}_0 dx - \iint_{Q_t} (\mathbf{w}_t)_x \cdot \mathbf{b} dx dt \\
&= - \int_{\Omega} \mathbf{w}_{0x} \cdot \mathbf{b}_0 dx - \int_{\Omega} \mathbf{w} \cdot \mathbf{b}_x dx + \iint_{Q_t} \mathbf{w}_t \cdot \mathbf{b}_x dx dt,
\end{aligned}$$

which together with Lemmas 2.1 and 2.2 and (2.25)₂ gives

$$\begin{aligned}
\iint_{Q_t} \mathbf{w}_x \cdot \mathbf{b}_t dx dt &\leq C + \frac{\nu}{4} \int_{\Omega} |\mathbf{b}_x|^2 dx + \left(\iint_{Q_t} |\mathbf{b}_x|^2 dx dt \right)^{1/2} \left(\iint_{Q_t} |\mathbf{w}_t|^2 dx dt \right)^{1/2} \\
&\leq C + \frac{\nu}{4} \int_{\Omega} |\mathbf{b}_x|^2 dx + C \left(\iint_{Q_t} |\mathbf{w}_t|^2 dx dt \right)^{1/2} \\
&\leq C + \frac{\nu}{4} \int_{\Omega} |\mathbf{b}_x|^2 dx + C \left(\iint_{Q_t} u_{xx}^2 dx ds \right)^{1/2}.
\end{aligned} \tag{2.29}$$

On the other hand, we have by Cauchy-Schwarz's inequality, Lemma 2.1 and (2.24)

$$\begin{aligned}
& - \iint_{Q_t} (u\mathbf{b})_x \cdot \mathbf{b}_t dx dt \\
& \leq \frac{1}{2} \iint_{Q_t} |\mathbf{b}_t|^2 dx dt + \frac{1}{2} \iint_{Q_t} |(u\mathbf{b})_x|^2 dx ds \\
& \leq \frac{1}{2} \iint_{Q_t} |\mathbf{b}_t|^2 dx dt + C \int_0^t \|u^2\|_{L^\infty(\Omega)} \int_{\Omega} |\mathbf{b}_x|^2 dx ds + C \int_0^t \|u_x\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\mathbf{b}|^2 dx ds \\
& \leq \frac{1}{2} \iint_{Q_t} |\mathbf{b}_t|^2 dx dt + C \int_0^t \|u^2\|_{L^\infty(\Omega)} \int_{\Omega} |\mathbf{b}_x|^2 dx ds + C \left(\iint_{Q_t} u_{xx}^2 dx ds \right)^{1/2}.
\end{aligned} \tag{2.30}$$

Substituting them into (2.28) and using Gronwall's inequality, we complete the proof. \square

Now we can prove the following desired results.

Lemma 2.10. *Let (1.4) and (1.8) hold. Then $\|(u, \mathbf{b})\|_{L^\infty(Q_T)} \leq C$, and*

$$\begin{aligned}
& \sup_{0 < t < T} \int_{\Omega} (\rho_t^2 + u_x^2 + \theta^2) dx + \iint_{Q_T} (u_t^2 + u_{xx}^2 + \kappa \theta_x^2) dx dt \leq C, \\
& \sup_{0 < t < T} \int_{\Omega} |\mathbf{w}_x|^2 \omega^2 dx + \iint_{Q_T} (|u|^2 |\mathbf{w}_x|^2 + |\mathbf{w}_t|^2) dx dt \leq C, \\
& \sup_{0 < t < T} \int_{\Omega} |\mathbf{b}_x|^2 dx + \iint_{Q_T} (|\mathbf{b}_t|^2 + |\mathbf{b}_{xx}|^2 \omega^2) dx dt \leq C.
\end{aligned} \tag{2.31}$$

Proof. Rewrite the equation (1.1)₂ in the form

$$\sqrt{\rho} u_t - \frac{\lambda}{\sqrt{\rho}} u_{xx} = -\sqrt{\rho} u u_x - \gamma \sqrt{\rho} \theta_x - \frac{\gamma}{\sqrt{\rho}} \rho_x \theta - \frac{1}{\sqrt{\rho}} \mathbf{b} \cdot \mathbf{b}_x.$$

Using Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}
& \frac{\lambda}{2} \int_{\Omega} u_x^2 dx + \iint_{Q_t} (\rho u_t^2 + \lambda^2 \rho^{-1} u_{xx}^2) dx dt \\
& \leq \frac{\lambda}{2} \int_{\Omega} u_{0x}^2 dx + C \iint_{Q_t} (u^2 u_x^2 + \theta_x^2 + \rho_x^2 \theta^2 + |\mathbf{b} \cdot \mathbf{b}_x|^2) dx ds \\
& \leq C + C \int_0^t \|u^2\|_{L^\infty(\Omega)} \int_{\Omega} u_x^2 dx ds + C \iint_{Q_t} \theta_x^2 dx ds + C \int_0^t \|\theta^2\|_{L^\infty(\Omega)} \int_{\Omega} \rho_x^2 dx ds \\
& \leq C + C \int_0^t \|u^2\|_{L^\infty(\Omega)} \int_{\Omega} u_x^2 dx ds + C \iint_{Q_t} (\theta_x^2 + \theta^2) dx ds,
\end{aligned} \tag{2.32}$$

where we used (2.10) and $\int_0^t \|\theta^2\|_{L^\infty(\Omega)} ds \leq C \iint_{Q_t} (\theta^2 + \theta_x^2) dx ds$.

Our next step is to multiply (1.1)₅ by θ and integrate over Q_t . We have

$$\frac{1}{2} \int_{\Omega} \rho \theta^2 dx + \iint_{Q_t} \kappa \theta_x^2 dx ds = - \iint_{Q_t} p u_x \theta dx ds + \iint_{Q_t} \theta \mathcal{Q} dx ds. \quad (2.33)$$

By Cauchy-Schwarz's inequality and (2.6), we obtain

$$\begin{aligned} -\gamma \iint_{Q_t} \rho \theta^2 u_x dx ds &\leq C \iint_{Q_t} \theta^2 dx ds + C \iint_{Q_t} \theta^2 u_x^2 dx ds \\ &\leq C + C \int_0^t \|\theta\|_{L^\infty(\Omega)}^2 \int_{\Omega} u_x^2 dx ds. \end{aligned}$$

On the other hand, we have by Lemmas 2.2, 2.6 and 2.9

$$\begin{aligned} \iint_{Q_t} \theta \mathcal{Q} dx ds &\leq C \int_0^t \|\theta\|_{L^\infty(\Omega)} \left\{ \int_{\Omega} (u_x^2 + \mu |\mathbf{w}_x|^2 dx + |\mathbf{b}_x|^2) dx \right\} ds \\ &\leq C + C \int_0^t \|\theta\|_{L^\infty(\Omega)} \int_{\Omega} u_x^2 dx ds + C \left(\iint_{Q_t} u_{xx}^2 dx ds \right)^{1/2}. \end{aligned}$$

Inserting them into (2.33) yields

$$\begin{aligned} \int_{\Omega} \theta^2 dx + \iint_{Q_t} \kappa \theta_x^2 dx ds \\ \leq C + C \int_0^t [\|\theta\|_{L^\infty(\Omega)} + \|\theta\|_{L^\infty(\Omega)}^2] \int_{\Omega} u_x^2 dx ds + C \left(\iint_{Q_t} u_{xx}^2 dx ds \right)^{1/2}. \end{aligned} \quad (2.34)$$

Plugging it into (2.32) gives

$$\begin{aligned} \int_{\Omega} u_x^2 dx + \iint_{Q_t} (u_t^2 + u_{xx}^2) dx dt \\ \leq C + C \int_0^t [\|u^2\|_{L^\infty(\Omega)} + \|\theta\|_{L^\infty(\Omega)} + \|\theta\|_{L^\infty(\Omega)}^2] \int_{\Omega} u_x^2 dx ds + C \left(\iint_{Q_t} u_{xx}^2 dx ds \right)^{1/2} \\ \leq C + C \int_0^t [\|u^2\|_{L^\infty(\Omega)} + \|\theta\|_{L^\infty(\Omega)} + \|\theta\|_{L^\infty(\Omega)}^2] \int_{\Omega} u_x^2 dx ds + \frac{1}{2} \iint_{Q_t} u_{xx}^2 dx ds. \end{aligned}$$

By Gronwall's inequality and noticing (2.6), we have

$$\int_{\Omega} u_x^2 dx + \iint_{Q_t} (u_t^2 + u_{xx}^2) dx dt \leq C.$$

Consequently, (2.31) follows from (2.34) and Lemmas 2.7-2.9. The proof is complete. \square

As a consequence of Lemma 2.10, we have

$$\begin{aligned} \iint_{Q_T} u_x^6 dx dt &\leq C \int_0^T \|u_x\|_{L^\infty(\Omega)}^4 \int_{\Omega} u_x^2 dx dt \leq C \int_0^T \|u_x\|_{L^\infty(\Omega)}^4 dt \\ &\leq C \int_0^T \left(\int_{\Omega} |u_x| |u_{xx}| dx \right)^2 dt \leq C \int_0^T \left(\int_{\Omega} u_x^2 dx \right) \left(\int_{\Omega} u_{xx}^2 dx \right) dt \\ &\leq C. \end{aligned} \quad (2.35)$$

Lemma 2.11. *Let (1.4) and (1.8) hold. Then $\|\mathbf{w}\|_{L^\infty(Q_T)} \leq C$. Moreover, it holds*

$$\sup_{0 < t < T} \int_{\Omega} |\mathbf{w}_x| dx \leq C.$$

Proof. Set $\mathbf{z} = \mathbf{w}_x$. Differentiating (2.16) in x gives

$$\mathbf{z}_t = \left(\frac{\mu}{\rho} \mathbf{z}_x \right)_x - (u\mathbf{z})_x + \left(\frac{\mathbf{b}_x}{\rho} \right)_x. \quad (2.36)$$

Denote $\Phi_\epsilon(\cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ for $\epsilon \in (0, 1)$ by

$$\Phi_\epsilon(\xi) = \sqrt{\epsilon^2 + |\xi|^2}, \quad \forall \xi \in \mathbb{R}^2.$$

Observe that Φ_ϵ has the properties

$$\begin{cases} |\xi| \leq |\Phi_\epsilon(\xi)| \leq |\xi| + \epsilon, & \forall \xi \in \mathbb{R}^2, \\ |\nabla_\xi \Phi_\epsilon(\xi)| \leq 1, & \forall \xi \in \mathbb{R}^2, \\ 0 \leq \xi \cdot \nabla_\xi \Phi_\epsilon(\xi) \leq \Phi_\epsilon(\xi), & \forall \xi \in \mathbb{R}^2, \\ \eta D_\xi^2 \Phi_\epsilon(\xi) \eta^\top \geq 0, & \forall \xi, \eta \in \mathbb{R}^2, \\ \lim_{\epsilon \rightarrow 0^+} \Phi_\epsilon(\xi) = |\xi|, & \forall \xi \in \mathbb{R}^2, \end{cases} \quad (2.37)$$

where ξ^\top stands for the transpose of the vector $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, and $D_\xi^2 g$ is the Hessian matrix of the function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ which is defined by

$$D_\xi^2 g(\xi) = \begin{pmatrix} g_{\xi_1 \xi_1} & g_{\xi_1 \xi_2} \\ g_{\xi_2 \xi_1} & g_{\xi_2 \xi_2} \end{pmatrix}.$$

Multiplying (2.36) by $\nabla_\xi \Phi_\epsilon(\mathbf{z})$ and integrating over Q_t , we have

$$\begin{aligned} & \int_{\Omega} \Phi_\epsilon(\mathbf{z}) dx - \int_{\Omega} \Phi_\epsilon(\mathbf{w}_{0x}) dx \\ &= -\mu \iint_{Q_t} \frac{1}{\rho} \mathbf{z}_x D_\xi^2 \Phi_\epsilon(\mathbf{z}) (\mathbf{z}_x)^\perp dx ds - \iint_{Q_t} (u\mathbf{z})_x \cdot \nabla_\xi \Phi_\epsilon(\mathbf{z}) dx ds \\ & \quad + \iint_{Q_t} \left(\frac{\mathbf{b}_x}{\rho} \right)_x \cdot \nabla_\xi \Phi_\epsilon(\mathbf{z}) dx ds + \mu \int_0^t \frac{\mathbf{z}_x \cdot \nabla_\xi \Phi_\epsilon(\mathbf{z})}{\rho} \Big|_{x=0}^{x=1} ds =: \sum_{j=1}^4 E_j. \end{aligned} \quad (2.38)$$

From (2.37)₄ it follows that

$$E_1 \leq 0.$$

To estimate E_2 , we observe by (2.37)₃

$$\begin{aligned} E_2 &= - \iint_{Q_t} (u\mathbf{z}_x + u_x \mathbf{z}) \cdot \nabla_\xi \Phi_\epsilon(\mathbf{z}) dx ds \\ &= \iint_{Q_t} (u_x \Phi_\epsilon(\mathbf{z}) - u_x \mathbf{z} \cdot \nabla_\xi \Phi_\epsilon(\mathbf{z})) dx ds \\ &\leq C \int_0^t \|u_x\|_{L^\infty(\Omega)} \int_{\Omega} \Phi_\epsilon(\mathbf{z}) dx ds. \end{aligned}$$

As to E_3 , utilizing the equation (1.1)₄ yields

$$\begin{aligned}
E_3 &= \iint_{Q_t} \frac{\mathbf{b}_{xx} \cdot \nabla_\xi \Phi_\epsilon(\mathbf{z})}{\rho} dx ds - \iint_{Q_t} \frac{\mathbf{b}_x \cdot \nabla_\xi \Phi_\epsilon(\mathbf{z})}{\rho^2} \rho_x dx ds \\
&= \frac{1}{\nu} \iint_{Q_t} \frac{[\mathbf{b}_t + (u\mathbf{b})_x - \mathbf{z}] \cdot \nabla_\xi \Phi_\epsilon(\mathbf{z})}{\rho} dx ds - \iint_{Q_t} \frac{\mathbf{b}_x \cdot \nabla_\xi \Phi_\epsilon(\mathbf{z})}{\rho^2} \rho_x dx ds \\
&\leq C \iint_{Q_t} [|\mathbf{b}_t| + |(u\mathbf{b})_x| + |\rho_x| |\mathbf{b}_x|] dx ds \leq C,
\end{aligned}$$

where we used (2.37)₂-(2.37)₃ and Lemmas 2.4 and 2.10.

It remains to estimate E_4 . From (2.16), we have

$$\left| \frac{\mu}{\rho(a, t)} \mathbf{z}_x(a, t) \right| = \left| \mathbf{w}_t(a, t) - \frac{\mathbf{b}_x(a, t)}{\rho(a, t)} \right| \leq C + C |\mathbf{b}_x(a, t)|, \quad \text{where } a = 0 \text{ or } a = 1. \quad (2.39)$$

On the other hand, we first integrate (1.1)₄ from a to $y \in [0, 1]$ in x , and then integrate the resulting equation over $(0, 1)$ in y , so that

$$\mathbf{b}_x(a, t) = -\frac{1}{\nu} \left\{ \int_0^1 \int_a^y \mathbf{b}_t(x, t) dx dy + \int_0^1 (u\mathbf{b} - \mathbf{w})(y, t) dy + \mathbf{w}(a, t) \right\},$$

so it follows from Lemmas 2.1 and 2.10 that

$$\int_0^T |\mathbf{b}_x(a, t)|^2 dt \leq C.$$

Thus one derives from (2.39) that

$$\int_0^T \left| \frac{\mu}{\rho(a, t)} \mathbf{z}_x(a, t) \right| dt \leq C + C \int_0^T |\mathbf{b}_x(a, t)| dt \leq C,$$

therefore

$$E_4 \leq C \int_0^T \left\{ \left| \frac{\mu}{\rho(1, t)} \mathbf{z}_x(1, t) \right| + \left| \frac{\mu}{\rho(0, t)} \mathbf{z}_x(0, t) \right| \right\} dt \leq C.$$

Substituting the above results in (2.38) and utilizing Gronwall's inequality, we get

$$\int_\Omega \Phi_\epsilon(\mathbf{z}) dx \leq C + \int_\Omega \Phi_\epsilon(\mathbf{w}_{0x}) dx.$$

Passing to the limit as $\epsilon \rightarrow 0$ yields

$$\int_\Omega |\mathbf{w}_x| dx \leq C.$$

This and $\int_\Omega |\mathbf{w}|^2 dx \leq C$ imply that $|\mathbf{w}| \leq C$. The proof is complete. \square

Lemma 2.12. *Let (1.4) and (1.8) hold. Then*

$$\int_0^T \|\mathbf{b}_x\|_{L^\infty(\Omega)}^2 dt \leq C.$$

Proof. For any fixed $z \in [0, 1]$, we first integrate (1.1)₄ from z to $y \in [0, 1]$ in x , and then integrate the resulting equation over $(0, 1)$ in y , so that

$$\mathbf{b}_x(z, t) = -\frac{1}{\nu} \left\{ \int_0^1 \int_z^y \mathbf{b}_t(x, t) dx dy + \int_0^1 (u\mathbf{b} - \mathbf{w})(y, t) dy - (u\mathbf{b} - \mathbf{w})(z, t) \right\},$$

which together with Lemmas 2.10 and 2.11 implies the desired result. The proof is complete. \square

Combining Lemmas 2.10-2.12, we have

$$\begin{aligned} \iint_{Q_T} |\mathbf{b}_x| |\mathbf{b}_{xx}| dx dt &= \frac{1}{\nu} \iint_{Q_T} |\mathbf{b}_x| |\mathbf{b}_t + (u\mathbf{b})_x - \mathbf{w}_x| dx dt \\ &\leq C + C \int_0^T \|\mathbf{b}_x\|_{L^\infty(\Omega)} \int_\Omega |\mathbf{w}_x| dx dt \leq C, \end{aligned} \quad (2.40)$$

and

$$\iint_{Q_T} |\mathbf{b}_x|^4 dx dt \leq C \int_0^T \|\mathbf{b}_x\|_{L^\infty(\Omega)}^2 \int_\Omega |\mathbf{b}_x|^2 dx dt \leq C. \quad (2.41)$$

Now some results in Lemmas 2.6 and 2.10 can be improved as follows.

Lemma 2.13. *Let (1.4) and (1.8) hold. Then*

$$\begin{aligned} \sqrt{\mu} \sup_{0 < t < T} \int_\Omega |\mathbf{w}_x|^2 dx + \mu^{3/2} \iint_{Q_T} |\mathbf{w}_{xx}|^2 dx dt &\leq C, \\ \sup_{0 < t < T} \int_\Omega |\mathbf{w}_x|^2 \omega dx + \iint_{Q_T} (\mu |\mathbf{w}_{xx}|^2 + |\mathbf{b}_{xx}|^2) \omega dx dt &\leq C. \end{aligned}$$

Proof. For the first estimate, we can use an argument similar to Lemma 2.6 to finish the proof. The key is to deal with the term $-\mu \iint_{Q_t} \frac{1}{\rho} \mathbf{b}_x \cdot \mathbf{w}_{xx} dx ds$ in (2.17).

By integrating by parts and using Cauchy-Schwarz's inequality, we have

$$\begin{aligned} &-\mu \iint_{Q_t} \frac{1}{\rho} \mathbf{b}_x \cdot \mathbf{w}_{xx} dx ds \\ &= \mu \iint_{Q_t} \frac{\mathbf{b}_{xx} \cdot \mathbf{w}_x}{\rho} dx ds - \mu \iint_{Q_t} \frac{\mathbf{b}_x \cdot \mathbf{w}_x}{\rho^2} \rho_x dx ds - \mu \int_0^T \frac{\mathbf{b}_x \cdot \mathbf{w}_x}{\rho} \Big|_{x=0}^{x=1} ds \\ &\leq C\mu \iint_{Q_t} |\mathbf{b}_{xx}|^2 dx ds + C\mu \iint_{Q_t} |\mathbf{w}_x|^2 dx ds + \mu \iint_{Q_t} |\mathbf{b}_x|^2 \rho_x^2 dx ds \\ &\quad + C\mu \left(\int_0^t \|\mathbf{b}_x\|_{L^\infty(\Omega)}^2 ds \right)^{1/2} \left(\int_0^t \|\mathbf{w}_x\|_{L^\infty(\Omega)}^2 ds \right)^{1/2} \\ &\leq C\mu + C\mu \iint_{Q_t} |\mathbf{b}_{xx}|^2 dx ds + C\mu \iint_{Q_t} |\mathbf{w}_x|^2 dx ds + C\mu \left(\int_0^t \|\mathbf{w}_x\|_{L^\infty(\Omega)}^2 ds \right)^{1/2}, \end{aligned} \quad (2.42)$$

where we used the fact by Lemmas 2.4 and 2.12

$$\iint_{Q_t} |\mathbf{b}_x|^2 |\rho_x|^2 dx ds \leq \int_0^t \|\mathbf{b}_x\|_{L^\infty(\Omega)}^2 \int_\Omega \rho_x^2 dx ds \leq C.$$

By (2.18), we obtain

$$\begin{aligned}
& \mu \left(\int_0^t \|\mathbf{w}_x\|_{L^\infty(\Omega)}^2 ds \right)^{1/2} \\
& \leq C\mu + C\mu^{1/4} \left(\mu \iint_{Q_t} |\mathbf{w}_x|^2 dx ds \right)^{1/4} \left(\mu^2 \iint_{Q_t} |\mathbf{w}_{xx}|^2 dx ds \right)^{1/4} \\
& \leq C\sqrt{\mu} + \frac{C\mu}{\epsilon} \iint_{Q_t} |\mathbf{w}_x|^2 dx ds + \epsilon\mu^2 \iint_{Q_t} \frac{1}{\rho} |\mathbf{w}_{xx}|^2 dx ds, \quad \forall \epsilon > 0.
\end{aligned} \tag{2.43}$$

It remains to show the estimate

$$\iint_{Q_t} |\mathbf{b}_{xx}|^2 dx ds \leq C + C \iint_{Q_t} |\mathbf{w}_x|^2 dx ds. \tag{2.44}$$

Multiplying (1.1)₄ by \mathbf{b}_{xx} and integrating over Q_t , we have

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} |\mathbf{b}_x|^2 dx + \nu \iint_{Q_t} |\mathbf{b}_{xx}|^2 dx ds \\
& = \frac{1}{2} \int_{\Omega} |\mathbf{b}_{0x}|^2 dx + \iint_{Q_t} u_x \mathbf{b} \cdot \mathbf{b}_{xx} dx ds - \frac{1}{2} \iint_{Q_t} u_x |\mathbf{b}_x|^2 dx ds - \iint_{Q_t} \mathbf{w}_x \cdot \mathbf{b}_{xx} dx ds \\
& \leq C + \frac{\nu}{2} \iint_{Q_t} |\mathbf{b}_{xx}|^2 dx ds + C \iint_{Q_t} |\mathbf{w}_x|^2 dx ds + C \int_0^t \|u_x\|_{L^\infty(\Omega)} \int_{\Omega} |\mathbf{b}_x|^2 dx ds,
\end{aligned}$$

where we used Lemma 2.10. Thus, (2.44) follows from Gronwall's inequality.

Inserting the above estimates into (2.42) and taking a small $\epsilon > 0$, we have

$$-\mu \iint_{Q_t} \frac{1}{\rho} \mathbf{b}_x \cdot \mathbf{w}_{xx} dx dt \leq C\sqrt{\mu} + C\mu \iint_{Q_t} |\mathbf{w}_x|^2 dx dt + \frac{\mu^2}{4} \iint_{Q_t} \frac{1}{\rho} |\mathbf{w}_{xx}|^2 dx ds.$$

Then, an argument similar to Lemma 2.6 leads to

$$\begin{aligned}
& \mu \int_{\Omega} |\mathbf{w}_x|^2 dx + \mu^2 \iint_{Q_t} |\mathbf{w}_{xx}|^2 dx ds \\
& \leq C\sqrt{\mu} + C \int_0^t (1 + \|u_x\|_{L^\infty(\Omega)}) \left(\mu \int_{\Omega} |\mathbf{w}_x|^2 dx \right) ds.
\end{aligned}$$

So the first estimate of this lemma follows from Gronwall's inequality and (2.13).

The second estimate can be proved by the arguments similar to Lemma 2.7 and (2.25)₁ and in terms of the first estimate and Lemmas 2.10-2.12. In fact, this can be done by using ω instead of ω^2 in (2.19) and (2.26) and noticing the following facts:

$$\begin{aligned}
& \mu \iint_{Q_T} |\mathbf{w}_x \cdot \mathbf{w}_{xx}| dx dt \leq C\sqrt{\mu} \iint_{Q_T} |\mathbf{w}_x|^2 dx dt + C\mu^{3/2} \iint_{Q_T} |\mathbf{w}_{xx}|^2 dx dt \leq C, \\
& \iint_{Q_T} |\mathbf{b}_x \cdot \mathbf{w}_x| dx dt \leq C \int_0^T \|\mathbf{b}_x\|_{L^\infty(\Omega)} \int_{\Omega} |\mathbf{w}_x| dx dt \leq C.
\end{aligned}$$

The proof is complete. \square

As a consequence of Lemma 2.13 and (2.43), we also have

$$\mu^{3/2} \iint_{Q_T} |\mathbf{w}_x|^4 dx dt \leq C \mu \int_0^T \|\mathbf{w}_x\|_{L^\infty(\Omega)}^2 \left(\sqrt{\mu} \int_{Q_T} |\mathbf{w}_x|^2 dx \right) dt \leq C. \quad (2.45)$$

Based on the above lemmas, we can bound the temperature θ in a direct way.

Lemma 2.14. *Let (1.4) and (1.8) hold. Then $\theta \leq C$.*

Proof. Rewrite the equation (1.1)₄ in the form

$$\theta_t = a(x, t)\theta_{xx} + b(x, t)\theta_x + c(x, t)\theta + f(x, t), \quad (2.46)$$

where

$$a = \rho^{-1}\kappa, \quad b = \rho^{-1}\kappa_x - u, \quad c = -\gamma u_x, \quad f = \rho^{-1}(\lambda u_x^2 + \mu |\mathbf{w}_x|^2 + \nu |\mathbf{b}_x|^2).$$

Set $z = \theta_x$. Differentiating the equation (2.46) in x yields

$$z_t = (az_x)_x + (bz)_x + cz + c_x\theta + f_x. \quad (2.47)$$

For $\epsilon \in (0, 1)$, denote $\varphi_\epsilon : \mathbb{R} \rightarrow \mathbb{R}^+$ by $\varphi_\epsilon(s) = \sqrt{s^2 + \epsilon^2}$. Simple calculations show that

$$\begin{cases} \varphi'_\epsilon(0) = 0, & |\varphi'_\epsilon(s)| \leq 1, & \varphi''_\epsilon(s) \geq 0, & |s\varphi''_\epsilon(s)| \leq 1, \\ \lim_{\epsilon \rightarrow 0} \varphi_\epsilon(s) = |s|, & \lim_{\epsilon \rightarrow 0} s\varphi''_\epsilon(s) = 0. \end{cases}$$

Multiplying (2.47) by $\varphi'_\epsilon(z)$, integrating over Q_t , and noticing $\varphi'_\epsilon(z)|_{x=0,1} = \varphi'_\epsilon(\theta_x)|_{x=0,1} = 0$, we have

$$\begin{aligned} \int_{\Omega} \varphi_\epsilon(z) dx - \int_{\Omega} \varphi_\epsilon(\theta_{0x}) dx ds &= - \iint_{Q_t} a \varphi''_\epsilon(z) z_x^2 dx ds - \iint_{Q_t} b z z_x \varphi''_\epsilon(z) dx ds \\ &\quad + \iint_{Q_t} (cz + c_x \theta + f_x) \varphi'_\epsilon(z) dx ds, \end{aligned}$$

and then, we obtain by $\varphi''_\epsilon(s) \geq 0$ and $|\varphi'_\epsilon(s)| \leq 1$

$$\begin{aligned} &\int_{\Omega} \varphi_\epsilon(z) dx - \int_{\Omega} \varphi_\epsilon(\theta_{0x}) dx \\ &\leq \iint_{Q_t} |bz_x| |z \varphi''_\epsilon(z)| dx ds + \iint_{Q_t} (|cz| + |c_x \theta| + |f_x|) dx ds. \end{aligned} \quad (2.48)$$

Recalling $|s\varphi''_\epsilon(s)| \leq 1$ and $s\varphi''_\epsilon(s) \rightarrow 0$ as $\epsilon \rightarrow 0$ and using Lebesgue's dominated convergence theorem, we obtain

$$\lim_{\epsilon \rightarrow 0} \iint_{Q_T} |bz_x| |z \varphi''_\epsilon(z)| dx dt = 0.$$

Thus, passing to the limit as $\epsilon \rightarrow 0$ in (2.48) and using $\lim_{\epsilon \rightarrow 0} \varphi_\epsilon(s) = |s|$, we have

$$\int_{\Omega} |\theta_x| dx \leq \int_{\Omega} |\theta_{0x}| dx + \iint_{Q_t} (|c\theta_x| + |c_x \theta| + |f_x|) dx ds. \quad (2.49)$$

By Lemma 2.10, we have

$$\begin{aligned}\iint_{Q_T} |c_x \theta| dx dt &\leq C \left(\iint_{Q_T} u_{xx}^2 dx dt \right)^{1/2} \left(\iint_{Q_T} \theta^2 dx dt \right)^{1/2} \leq C, \\ \iint_{Q_T} |c \theta_x| dx dt &\leq C \left(\iint_{Q_T} u_x^2 dx dt \right)^{1/2} \left(\iint_{Q_T} \theta_x^2 dx dt \right)^{1/2} \leq C.\end{aligned}$$

By Cauchy-Schwarz's inequality, (2.40), Lemmas 2.4 and 2.10, (2.35), (2.41) and (2.45), we obtain

$$\begin{aligned}\iint_{Q_T} |f_x| dx dt &\leq C \iint_{Q_T} (|u_x| |u_{xx}| + \mu |\mathbf{w}_x \cdot \mathbf{w}_{xx}| + |\mathbf{b}_x \cdot \mathbf{b}_{xx}|) dx dt \\ &\quad + C \iint_{Q_T} (u_x^2 + \mu |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2) |\rho_x| dx dt \\ &\leq C + C \iint_{Q_T} (u_x^2 + u_{xx}^2 + \sqrt{\mu} |\mathbf{w}_x|^2 + \mu^{3/2} |\mathbf{w}_{xx}|^2) dx dt \\ &\quad + C \iint_{Q_T} (u_x^4 + \mu^2 |\mathbf{w}_x|^4 + |\mathbf{b}_x|^4) dx dt + C \iint_{Q_T} \rho_x^2 dx dt \leq C.\end{aligned}$$

Substituting the above estimates into (2.49) yields

$$\int_{\Omega} |\theta_x| dx \leq C,$$

which together with $\int_{\Omega} \theta dx \leq C$ implies the desired result. The proof is complete. \square

By means of the bounds of θ , we can obtain easily the following estimates.

Lemma 2.15. *Let (1.4) and (1.8) hold. Then*

$$\sup_{0 < t < T} \int_{\Omega} \theta_x^2 dx + \iint_{Q_T} (\theta_t^2 + \theta_{xx}^2) dx dt \leq C.$$

Proof. Rewrite the equation (1.1)₅ in the form

$$\rho \theta_t - (\kappa \theta_x)_x = \mathcal{Q} - \rho u \theta_x - \gamma \rho \theta u_x := f. \quad (2.50)$$

We first estimate $\|f\|_{L^2(Q_T)}$. By (2.35), (2.41), (2.45) and Lemmas 2.10 and 2.14, we have

$$\iint_{Q_T} f^2 dx dt \leq C \iint_{Q_T} (u_x^4 + \mu^2 |\mathbf{w}_x|^4 + \nu^2 |\mathbf{b}_x|^4 + \rho^2 u^2 \theta_x^2 + \rho^2 u_x^2 \theta^2) dx dt \leq C. \quad (2.51)$$

Multiplying (2.50) by $\kappa \theta_t$ and integrating over Q_t , we have

$$\iint_{Q_t} \rho \kappa \theta_t^2 dx dt + \iint_{Q_t} \kappa \theta_x (\kappa \theta_t)_x dx dt = \iint_{Q_t} f \kappa \theta_t dx dt. \quad (2.52)$$

Observe that

$$(\kappa \theta_t)_x = (\kappa \theta_x)_t + \kappa_\rho \rho_x \theta_t + \kappa_\rho \theta_x (\rho_x u + \rho u_x),$$

so that

$$\begin{aligned} \iint_{Q_t} \kappa \theta_x (\kappa \theta_t)_x dx dt &= \frac{1}{2} \int_{\Omega} \kappa^2 \theta_x^2 dx - \frac{1}{2} \int_{\Omega} \kappa^2 (\rho_0, \theta_0) \theta_{0x}^2 dx \\ &\quad + \iint_{Q_t} \left[\kappa \kappa_{\rho} \rho_x \theta_x \theta_t + \kappa \kappa_{\rho} \theta_x^2 (\rho_x u + \rho u_x) \right] dx dt, \end{aligned}$$

and substitute it into (2.52) to yield

$$\begin{aligned} &\iint_{Q_t} \rho \kappa \theta_t^2 dx dt + \int_{\Omega} \kappa^2 \theta_x^2 dx \\ &\leq C - 2 \iint_{Q_t} \left[\kappa \kappa_{\rho} \rho_x \theta_x \theta_t + \kappa \kappa_{\rho} \theta_x^2 (\rho_x u + \rho u_x) - f \kappa \theta_t \right] dx dt. \end{aligned} \quad (2.53)$$

By the estimates $C^{-1} \leq \rho, \theta \leq C$ and (1.4), we have $\kappa_1 \leq \kappa \leq C, |\kappa_{\rho}| \leq C$. By Young's inequality, (2.10), (2.51) and Lemma 2.10, we obtain

$$\begin{aligned} &-2 \iint_{Q_t} \left[\kappa \kappa_{\rho} \rho_x \theta_x \theta_t + \kappa \kappa_{\rho} \theta_x^2 (\rho_x u + \rho u_x) - f \kappa \theta_t \right] dx dt \\ &\leq C + \frac{1}{4} \iint_{Q_t} \rho \kappa \theta_t^2 dx dt + C \iint_{Q_t} (\kappa \theta_x)^2 (\rho_x^2 + |\rho_x| + |u_x|) dx ds \\ &\leq C + \frac{1}{4} \iint_{Q_t} \rho \kappa \theta_t^2 dx dt + C \int_0^t \|\kappa \theta_x\|_{L^\infty(\Omega)}^2 ds. \end{aligned} \quad (2.54)$$

Now we are ready to deal with the second integral on right-hand side of (2.54). By the embedding $W^{1,1}(\Omega) \hookrightarrow L^\infty(\Omega)$ and Young's inequality, we have

$$\begin{aligned} \int_0^t \|\kappa \theta_x\|_{L^\infty(\Omega)}^2 ds &\leq \iint_{Q_t} |\kappa \theta_x|^2 dx ds + 2 \iint_{Q_t} |\kappa \theta_x| |(\kappa \theta_x)_x| dx ds \\ &\leq \frac{C}{\epsilon} + \frac{\epsilon}{2} \iint_{Q_t} |(\kappa \theta_x)_x|^2 dx ds, \quad \forall \epsilon > 0, \end{aligned}$$

which together with (2.50) gives

$$\int_0^t \|\kappa \theta_x\|_{L^\infty(\Omega)}^2 ds \leq \frac{C}{\epsilon} + \epsilon \iint_{Q_t} (\rho^2 \theta_t^2 + f^2) dx dt.$$

Plugging it into (2.54), taking a small $\epsilon > 0$ and using (2.51), we obtain

$$-2 \iint_{Q_t} \left[\kappa \kappa_{\rho} \rho_x \theta_x \theta_t + \kappa \kappa_{\rho} \theta_x^2 (\rho_x u + \rho u_x) - f \kappa \theta_t \right] dx dt \leq C + \frac{1}{2} \iint_{Q_t} \rho \kappa \theta_t^2 dx dt,$$

from which, (1.4) and (2.53) it follows that

$$\sup_{0 < t < T} \int_{\Omega} \theta_x^2 dx + \iint_{Q_T} \theta_t^2 dx dt \leq C. \quad (2.55)$$

By (2.55) and Lemma 2.14, one can derive easily from (1.1)₅ that $\|\theta_{xx}\|_{L^2(Q_T)} \leq C$. The proof is complete. \square

Due to Lemma 2.15, an argument similar to (2.35) yields

$$\iint_{Q_T} \theta_x^6 dx dt \leq C. \quad (2.56)$$

Thus, all the estimates appearing in Theorem 1.1 are proved.

2.2 Proof of Theorem 1.1(ii)

By an argument similar to (2.10)₂, one has

$$\begin{aligned} \|(u, \mathbf{b}, \theta)\|_{C^{1/2,1/4}(\overline{Q}_T)} &\leq C, \\ \|\mathbf{w}\|_{C^{1/2,1/4}([\delta, 1-\delta] \times [0, T])} &\leq C, \quad \forall \delta \in (0, (b-a)/2). \end{aligned} \quad (2.57)$$

From (2.10)₂, (2.35), (2.41), (2.56), (2.57) and Lemmas 2.2, 2.4, 2.10-2.15 it follows that there exist a subsequence $\mu_j \rightarrow 0$ and $(\overline{\rho}, \overline{u}, \overline{\mathbf{w}}, \overline{\mathbf{b}}, \overline{\theta}) \in \mathbb{F}$ such that the corresponding solution for problem (1.1)-(1.3) with $\mu = \mu_j$, still denoted by $(\rho, u, \mathbf{w}, \mathbf{b}, \theta)$, converges in the sense:

$$\begin{aligned} (\rho, u, \mathbf{b}, \theta) &\rightarrow (\overline{\rho}, \overline{u}, \overline{\mathbf{b}}, \overline{\theta}) \text{ strongly in } C^\alpha(\overline{Q}_T), \quad \forall \alpha \in (0, 1/4), \\ (\rho_t, \rho_x, u_x, \mathbf{b}_x, \theta_x) &\rightharpoonup (\overline{\rho}_t, \overline{\rho}_x, \overline{u}_x, \overline{\mathbf{b}}_x, \overline{\theta}_x) \text{ weakly} - * \text{ in } L^\infty(0, T; L^2(\Omega)), \\ (u_t, \mathbf{b}_t, \theta_t, u_{xx}, \theta_{xx}) &\rightharpoonup (\overline{u}_t, \overline{\mathbf{b}}_t, \overline{\theta}_t, \overline{u}_{xx}, \overline{\theta}_{xx}) \text{ weakly in } L^2(Q_T), \\ \mathbf{b}_{xx} &\rightharpoonup \overline{\mathbf{b}}_{xx} \text{ weakly in } L^2((a+\delta, b-\delta) \times (0, T)), \quad \forall \delta \in (0, (b-a)/2), \end{aligned}$$

and

$$\begin{aligned} \mathbf{w} &\rightarrow \overline{\mathbf{w}} \text{ strongly in } C^\alpha([a+\delta, b-\delta] \times [0, T]), \quad \forall \delta \in (0, (b-a)/2), \quad \alpha \in (0, 1/4), \\ \mathbf{w}_t &\rightharpoonup \overline{\mathbf{w}}_t \text{ weakly in } L^2(Q_T), \\ \mathbf{w}_x &\rightharpoonup \overline{\mathbf{w}}_x \text{ weakly} - * \text{ in } L^\infty(0, T; L^2(a+\delta, b-\delta)), \quad \forall \delta \in (0, (b-a)/2), \\ \mathbf{w} &\rightarrow \overline{\mathbf{w}} \text{ strongly in } L^r(Q_T), \quad \forall r \in [1, +\infty), \\ \sqrt{\mu} \|\mathbf{w}_x\|_{L^4(Q_T)} &\rightarrow 0. \end{aligned}$$

Next we show the strong convergence of $(u_x, \mathbf{b}_x, \theta_x)$ in $L^2(Q_T)$. Multiplying (1.1)₂ with $\mu = \mu_j$ by $(u - \overline{u})$ and integrating over Q_T , we have

$$\begin{aligned} &\lambda \iint_{Q_T} (u_x - \overline{u}_x)^2 dx dt + \lambda \iint_{Q_T} \overline{u}_x (u_x - \overline{u}_x) dx dt \\ &= - \iint_{Q_T} \left[(\rho u)_t + \left(\rho u^2 + \gamma \rho \theta + \frac{1}{2} |\mathbf{b}|^2 \right)_x \right] (u - \overline{u}) dx dt, \end{aligned}$$

which together with Lemmas 2.4, 2.10 and 2.14 implies that

$$u_x \rightarrow \overline{u}_x \text{ strongly in } L^2(Q_T) \text{ as } \mu_j \rightarrow 0.$$

Similarly, one has

$$(\mathbf{b}_x, \theta_x) \rightarrow (\overline{\mathbf{b}}_x, \overline{\theta}_x) \text{ strongly in } L^2(Q_T) \text{ as } \mu_j \rightarrow 0.$$

Furthermore, since from (2.35), (2.41) and (2.56), we have

$$\begin{aligned} (u_x, \theta_x) &\rightarrow (\overline{u}_x, \overline{\theta}_x) \text{ strongly in } L^{s_1}(Q_T) \text{ as } \mu_j \rightarrow 0, \quad \forall s_1 \in [1, 6), \\ \mathbf{b}_x &\rightarrow \overline{\mathbf{b}}_x \text{ strongly in } L^{s_2}(Q_T) \text{ as } \mu_j \rightarrow 0, \quad \forall s_2 \in [1, 4). \end{aligned}$$

Then, it is easy to check that $(\overline{\rho}, \overline{u}, \overline{\mathbf{w}}, \overline{\mathbf{b}}, \overline{\theta})$ satisfies (1.10).

On the other hand, one can see from Theorem 1.1(iii) that the limit problem (1.10) admits at most one solution in \mathbb{F} . Thus, the above convergence relations hold for any $\mu_j \rightarrow 0$. The proof of Theorem 1.1(ii) is then completed.

2.3 Proof of Theorem 1.1(iii)

The proof is divided into several steps among which the fourth step is the key that can be proved in terms of the boundary estimates of \mathbf{w}_x . For convenience, we set

$$\begin{aligned}\tilde{\rho} &= \rho - \bar{\rho}, \quad \tilde{u} = u - \bar{u}, \quad \tilde{\mathbf{w}} = \mathbf{w} - \bar{\mathbf{w}}, \quad \tilde{\mathbf{b}} = \mathbf{b} - \bar{\mathbf{b}}, \quad \tilde{\theta} = \theta - \bar{\theta}, \\ \mathbb{H}(t) &= \|(\tilde{\rho}, \tilde{u}, \tilde{\mathbf{w}}, \tilde{\mathbf{b}}, \tilde{\theta})\|_{L^2(\Omega)}^2, \\ D(t) &= 1 + \|(u_x, \mathbf{b}_x, \bar{u}_x, \bar{\mathbf{b}}_x, \bar{\theta}_x)\|_{L^\infty(\Omega)}^2 + \|(\bar{u}_t, \bar{\theta}_t, \bar{u}_x, \bar{\mathbf{b}}_x, \bar{\theta}_x)\|_{L^2(\Omega)}^2.\end{aligned}$$

Clearly, $D(t) \in L^1(0, T)$.

Step 1 We claim that

$$\int_{\Omega} \tilde{\rho}^2 dx \leq \epsilon \iint_{Q_t} \tilde{u}_x^2 dx ds + \frac{C}{\epsilon} \int_0^t D(s) \mathbb{H}(s) ds, \quad \forall \epsilon \in (0, 1). \quad (2.58)$$

From (1.1)₁ and (1.10)₁ it follows that

$$\tilde{\rho}_t = -(\rho \tilde{u} + \bar{u} \tilde{\rho})_x.$$

Multiplying it by $\tilde{\rho}$ and integrating over Q_t , we have by Young's inequality

$$\begin{aligned}\frac{1}{2} \int_{\Omega} \tilde{\rho}^2 dx &= - \iint_{Q_t} (\rho \tilde{u}_x \tilde{\rho} + \rho_x \tilde{u} \tilde{\rho}) dx ds - \frac{1}{2} \iint_{Q_t} \bar{u}_x \tilde{\rho}^2 dx ds \\ &\leq \frac{\epsilon}{4} \iint_{Q_t} \tilde{u}_x^2 dx ds + C \iint_{Q_t} \tilde{u}^2 \rho_x^2 dx ds \\ &\quad + \frac{C}{\epsilon} \int_0^t (1 + \|\bar{u}_x\|_{L^\infty(\Omega)}) \int_{\Omega} \tilde{\rho}^2 dx ds, \quad \forall \epsilon \in (0, 1).\end{aligned}$$

Since from (2.10), we have

$$C \iint_{Q_t} \tilde{u}^2 \rho_x^2 dx ds \leq C \int_0^t \|\tilde{u}\|_{L^\infty(\Omega)}^2 ds \leq \frac{\epsilon}{4} \iint_{Q_t} \tilde{u}_x^2 dx ds + \frac{C}{\epsilon} \iint_{Q_t} \tilde{u}^2 dx ds.$$

Thus, the claim (2.58) is proved.

Step 2 We claim that

$$\int_{\Omega} \tilde{u}^2 dx + \iint_{Q_t} \tilde{u}_x^2 dx ds \leq C \int_0^t D(s) \mathbb{H}(s) ds. \quad (2.59)$$

Using (1.1)₁ and (1.10)₁, we derive from (1.1)₂ and (1.10)₂ that

$$(\rho \tilde{u})_t + (\rho u \tilde{u})_x + \tilde{\rho} \bar{u}_t + (\rho u - \bar{\rho} \bar{u}) \bar{u}_x + \gamma(\rho \theta - \bar{\rho} \bar{\theta})_x + \frac{1}{2}(|\mathbf{b}|^2 - |\bar{\mathbf{b}}|^2)_x = \lambda \tilde{u}_{xx}.$$

Multiplying it by \tilde{u} and integrating over Q_t , we have

$$\begin{aligned}&\frac{1}{2} \int_{\Omega} \rho \tilde{u}^2 dx + \lambda \iint_{Q_t} \tilde{u}_x^2 dx ds \\ &= - \iint_{Q_t} \tilde{\rho} \bar{u}_t \tilde{u} dx ds - \iint_{Q_t} (\rho u - \bar{\rho} \bar{u}) \bar{u}_x \tilde{u} dx ds + \gamma \iint_{Q_t} (\rho \theta - \bar{\rho} \bar{\theta}) \tilde{u}_x dx ds \\ &\quad + \frac{1}{2} \iint_{Q_t} (|\mathbf{b}|^2 - |\bar{\mathbf{b}}|^2) \tilde{u}_x dx ds =: \sum_{i=1}^4 I_i.\end{aligned} \quad (2.60)$$

Observe that $\rho u - \bar{\rho} \bar{u} = \rho \tilde{u} + \bar{u} \tilde{\rho}$ and $\rho \theta - \bar{\rho} \bar{\theta} = \rho \tilde{\theta} + \bar{\theta} \tilde{\rho}$. We have

$$\begin{aligned}
& I_1 + I_2 \\
& \leq C \int_0^t \|\tilde{u}\|_{L^\infty(\Omega)} \int_\Omega |\tilde{\rho}|(|\bar{u}_t| + |\bar{u}_x|) dt + C \iint_{Q_t} |\bar{u}_x| \tilde{u}^2 dx ds \\
& \leq C \int_0^t \left(\int_\Omega \tilde{\rho}^2 dx \right)^{1/2} \left(\int_\Omega (\bar{u}_t^2 + \bar{u}_x^2) dx \right)^{1/2} \|\tilde{u}\|_{L^\infty(\Omega)} dt + C \int_0^t \|\bar{u}_x\|_{L^\infty(\Omega)} \int_\Omega \tilde{u}^2 dx ds \quad (2.61) \\
& \leq C \int_0^t \left(\int_\Omega (\bar{u}_t^2 + \bar{u}_x^2) dx \right) \left(\int_\Omega \tilde{\rho}^2 dx \right) dt + C \int_0^t \|\tilde{u}\|_{L^\infty(\Omega)}^2 ds \\
& \quad + C \int_0^t \|\bar{u}_x\|_{L^\infty(\Omega)} \int_\Omega \tilde{u}^2 dx ds \leq C \int_0^t D(s) \mathbb{H}(s) ds + \frac{\lambda}{4} \iint_{Q_t} \tilde{u}_x^2 dx ds,
\end{aligned}$$

and

$$I_3 \leq \frac{\lambda}{4} \iint_{Q_t} \tilde{u}_x^2 dx ds + C \iint_{Q_t} (\tilde{\theta}^2 + \tilde{\rho}^2) dx ds. \quad (2.62)$$

Utilizing the estimates $\|(\mathbf{b}, \bar{\mathbf{b}})\|_{L^\infty(Q_T)} \leq C$, we have

$$I_4 \leq \frac{\lambda}{4} \iint_{Q_t} \tilde{u}_x^2 dx ds + C \iint_{Q_t} |\tilde{\mathbf{b}}|^2 dx ds. \quad (2.63)$$

Substituting (2.61)-(2.63) into (2.60) completes the proof to (2.59).

Step 3 We claim that

$$\begin{aligned}
& \int_\Omega \tilde{\theta}^2 dx + \iint_{Q_t} \tilde{\theta}_x^2 dx ds \\
& \leq C \sqrt{\mu} + \epsilon \iint_{Q_t} (\tilde{u}_x^2 + |\tilde{\mathbf{b}}_x|^2) dx ds + \frac{C}{\epsilon} \int_0^t D(s) \mathbb{H}(s) ds, \quad \forall \epsilon \in (0, 1).
\end{aligned} \quad (2.64)$$

From (1.1)₅ and (1.10)₅ it follows that

$$\begin{aligned}
& (\rho \tilde{\theta})_t + (\rho \tilde{\theta})_x + \tilde{\rho} \bar{\theta}_t + (\rho \tilde{u} + \bar{u} \tilde{\rho}) \bar{\theta}_x + \gamma \rho \theta \tilde{u}_x + \gamma (\rho \tilde{\theta} + \tilde{\rho} \bar{\theta}) \bar{u}_x = [\kappa(\rho, \theta) \tilde{\theta}_x]_x \\
& + [(\kappa(\rho, \theta) - \kappa(\bar{\rho}, \bar{\theta})) \bar{\theta}_x]_x + \lambda(u_x^2 - \bar{u}_x^2) + \mu |\mathbf{w}_x|^2 + \nu (|\mathbf{b}_x|^2 - |\bar{\mathbf{b}}_x|^2).
\end{aligned}$$

Multiplying it by $\tilde{\theta}$ and integrating over Q_t , we obtain

$$\begin{aligned}
& \frac{1}{2} \int_\Omega \rho \tilde{\theta}^2 dx + \iint_{Q_t} \kappa \tilde{\theta}_x^2 dx ds \\
& = - \iint_{Q_t} \tilde{\rho} \tilde{\theta} \bar{\theta}_t dx dt - \iint_{Q_t} (\rho \tilde{u} + \bar{u} \tilde{\rho}) \tilde{\theta} \bar{\theta}_x dx ds - \gamma \iint_{Q_t} \rho \theta \tilde{u}_x \tilde{\theta} dx ds \\
& \quad - \gamma \iint_{Q_t} \rho \bar{u}_x \tilde{\theta}^2 dx ds - \gamma \iint_{Q_t} \bar{\theta} \bar{u}_x \tilde{\rho} \tilde{\theta} dx ds - \iint_{Q_t} \bar{\theta}_x [\kappa(\rho, \theta) - \kappa(\bar{\rho}, \bar{\theta})] \tilde{\theta} dx ds \quad (2.65) \\
& \quad + \lambda \iint_{Q_t} (u_x + \bar{u}_x) \tilde{u}_x \tilde{\theta} dx ds + \mu \iint_{Q_t} |\mathbf{w}_x|^2 \tilde{\theta} dx ds \\
& \quad + \nu \iint_{Q_t} (|\mathbf{b}_x|^2 - |\bar{\mathbf{b}}_x|^2) \tilde{\theta} dx ds =: \sum_{i=1}^9 E_i.
\end{aligned}$$

By Hölder's inequality and Young's inequality, we have

$$\begin{aligned}
E_1 + E_2 + E_5 &\leq C \int_0^t \left(\int_{\Omega} (\tilde{\rho}^2 + \tilde{u}^2) dx \right)^{1/2} \left(\int_{\Omega} \tilde{\theta}^2 (\bar{\theta}_t^2 + \bar{\theta}_x^2 + \bar{u}_x^2) dx \right)^{1/2} dt \\
&\leq C \int_0^t \left(\int_{\Omega} (\bar{\theta}_t^2 + \bar{\theta}_x^2 + \bar{u}_x^2) dx \right)^{1/2} \left(\int_{\Omega} (\tilde{\rho}^2 + \tilde{u}^2) dx \right)^{1/2} \|\tilde{\theta}\|_{L^\infty(\Omega)} dt \\
&\leq C \int_0^t \left(\int_{\Omega} (\bar{\theta}_t^2 + \bar{\theta}_x^2 + \bar{u}_x^2) dx \right) \left(\int_{\Omega} (\tilde{\rho}^2 + \tilde{u}^2) dx \right) dt + \int_0^t \|\tilde{\theta}\|_{L^\infty(\Omega)}^2 ds \\
&\leq C \int_0^t D(s) \mathbb{H}(s) ds + \frac{\kappa_1}{4} \iint_{Q_t} \tilde{\theta}_x^2 dx ds + C \iint_{Q_t} \tilde{\theta}^2 dx ds \\
&\leq C \int_0^t D(s) \mathbb{H}(s) ds + \frac{\kappa_1}{4} \iint_{Q_t} \tilde{\theta}_x^2 dx ds.
\end{aligned}$$

By Young's inequality, we have

$$\begin{aligned}
&E_3 + E_4 + E_7 \\
&\leq \epsilon \iint_{Q_t} \tilde{u}_x^2 dx ds + \frac{C}{\epsilon} \int_0^t (1 + \|u_x\|_{L^\infty(\Omega)}^2 + \|\bar{u}_x\|_{L^\infty(\Omega)}^2) \int_{\Omega} \tilde{\theta}^2 dx ds \\
&\leq \epsilon \iint_{Q_t} \tilde{u}_x^2 dx ds + \frac{C}{\epsilon} \int_0^t D(s) \mathbb{H}(s) ds, \quad \forall \epsilon \in (0, 1).
\end{aligned}$$

By the mean value theorem and $C^{-1} \leq \rho, \bar{\rho}, \theta, \bar{\theta} \leq C$, we obtain

$$|\kappa(\rho, \theta) - \kappa(\bar{\rho}, \bar{\theta})| \leq C(|\tilde{\rho}| + |\tilde{\theta}|),$$

so

$$\begin{aligned}
E_6 &\leq \frac{\kappa_1}{4} \iint_{Q_t} \tilde{\theta}_x^2 dx ds + C \iint_{Q_t} |\bar{\theta}_x|^2 (\tilde{\rho}^2 + \tilde{\theta}^2) dx ds \\
&\leq \frac{\kappa_1}{4} \iint_{Q_t} \tilde{\theta}_x^2 dx ds + C \int_0^t \|\bar{\theta}_x\|_{L^\infty(\Omega)}^2 \int_{\Omega} (\tilde{\rho}^2 + \tilde{\theta}^2) dx ds \\
&\leq \frac{\kappa_1}{4} \iint_{Q_t} \tilde{\theta}_x^2 dx ds + C \int_0^t D(s) \mathbb{H}(s) ds.
\end{aligned}$$

By (2.45), we have

$$E_8 \leq C \iint_{Q_t} \tilde{\theta}^2 dx ds + C \mu^2 \iint_{Q_t} |\mathbf{w}_x|^4 dx ds \leq C \sqrt{\mu} + C \int_0^t D(s) \mathbb{H}(s) ds.$$

As to E_9 , we have by the relation: $|\mathbf{b}_x|^2 - |\bar{\mathbf{b}}_x|^2 = (\mathbf{b}_x + \bar{\mathbf{b}}_x) \cdot \tilde{\mathbf{b}}_x$

$$\begin{aligned}
E_9 &\leq \epsilon \iint_{Q_t} |\tilde{\mathbf{b}}_x|^2 dx ds + \frac{C}{\epsilon} \int_0^t (\|\mathbf{b}_x\|_{L^\infty(\Omega)}^2 + \|\bar{\mathbf{b}}_x\|_{L^\infty(\Omega)}^2) \int_{\Omega} \tilde{\theta}^2 dx ds \\
&\leq \epsilon \iint_{Q_t} |\tilde{\mathbf{b}}_x|^2 dx ds + \frac{C}{\epsilon} \int_0^t D(s) \mathbb{H}(s) ds, \quad \forall \epsilon \in (0, 1).
\end{aligned}$$

Substituting the results into (2.65) completes the proof to (2.64).

Step 4 We claim that

$$\int_{\Omega} |\tilde{\mathbf{w}}|^2 dx \leq C\sqrt{\mu} + \epsilon \iint_{Q_t} |\tilde{\mathbf{b}}_x|^2 dx ds + \frac{C}{\epsilon} \int_0^t D(s) \mathbb{H}(s) ds, \quad \forall \epsilon \in (0, 1). \quad (2.66)$$

From (1.1)₃ and (1.10)₃, we have

$$\rho \tilde{\mathbf{w}}_t + \rho u \tilde{\mathbf{w}}_x + \rho \tilde{u} \overline{\mathbf{w}}_x - \tilde{\mathbf{b}}_x + \frac{\tilde{\rho}}{\bar{\rho}} \overline{\mathbf{b}}_x = \mu \mathbf{w}_{xx}.$$

Multiplying it by $\tilde{\mathbf{w}}$ and integrating over Q_t , we have

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \rho |\tilde{\mathbf{w}}|^2 dx &= \mu \iint_{Q_t} \mathbf{w}_{xx} \cdot \tilde{\mathbf{w}} dx ds - \iint_{Q_t} \rho \tilde{u} \overline{\mathbf{w}}_x \cdot \tilde{\mathbf{w}} \\ &\quad + \iint_{Q_t} \tilde{\mathbf{b}}_x \cdot \tilde{\mathbf{w}} dx ds - \iint_{Q_t} \frac{\tilde{\rho}}{\bar{\rho}} \overline{\mathbf{b}}_x \cdot \tilde{\mathbf{w}} dx ds \\ &\leq C\mu^2 \iint_{Q_t} |\mathbf{w}_{xx}|^2 dx ds + C \iint_{Q_t} \tilde{u}^2 |\overline{\mathbf{w}}_x|^2 dx ds \\ &\quad + \frac{C}{\epsilon} \int_0^t D(s) \mathbb{H}(s) ds + \epsilon \iint_{Q_t} |\tilde{\mathbf{b}}_x|^2 dx ds, \quad \forall \epsilon \in (0, 1). \end{aligned} \quad (2.67)$$

Observe that

$$\begin{aligned} |\tilde{u}(x, t)| &= |u(x, t) - \bar{u}(x, t)| = \left| \int_0^x \tilde{u}_x dx \right| \leq \left(\int_0^1 \tilde{u}_x^2 dx \right)^{1/2} \omega^{1/2}(x), \quad \forall x \in [0, 1/2], \\ |\tilde{u}(x, t)| &= |u(x, t) - \bar{u}(x, t)| = \left| \int_x^1 \tilde{u}_x dx \right| \leq \left(\int_0^1 \tilde{u}_x^2 dx \right)^{1/2} \omega^{1/2}(x), \quad \forall x \in [1/2, 1]. \end{aligned}$$

We have

$$|\tilde{u}(x, t)|^2 \leq \left(\int_0^1 \tilde{u}_x^2 dx \right) \omega(x), \quad \forall (x, t) \in \overline{Q}_T,$$

which together with Lemma 2.13 and (2.59) gives

$$\begin{aligned} \iint_{Q_t} \tilde{u}^2 |\overline{\mathbf{w}}_x|^2 dx ds &\leq \int_0^t \left(\int_0^1 \tilde{u}_x^2 dx \right) \left(\int_{\Omega} |\overline{\mathbf{w}}_x|^2 \omega dx \right) ds \leq C \iint_{Q_t} \tilde{u}_x^2 dx ds \\ &\leq C \int_0^t D(s) \mathbb{H}(s) ds. \end{aligned}$$

Substituting it into (2.67) completes the proof to (2.66).

Step 5 We claim that

$$\int_{\Omega} |\tilde{\mathbf{b}}|^2 dx + \iint_{Q_t} |\tilde{\mathbf{b}}_x|^2 dx ds \leq C \int_0^t D(s) \mathbb{H}(s) ds. \quad (2.68)$$

From (1.1)₄ and (1.10)₄, we have

$$\tilde{\mathbf{b}}_t + (u \tilde{\mathbf{b}})_x + (\tilde{u} \overline{\mathbf{b}})_x - \tilde{\mathbf{w}}_x - \nu \tilde{\mathbf{b}}_{xx} = 0.$$

Multiplying it by $\tilde{\mathbf{b}}$ and integrating over Q_t yield

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\tilde{\mathbf{b}}|^2 dx + \nu \iint_{Q_t} |\tilde{\mathbf{b}}_x|^2 dx ds \\ &= -\frac{1}{2} \iint_{Q_t} u_x |\tilde{\mathbf{b}}|^2 dx ds + \iint_{Q_t} \tilde{u} \tilde{\mathbf{b}} \cdot \tilde{\mathbf{b}}_x dx ds - \iint_{Q_t} \tilde{\mathbf{b}}_x \cdot \tilde{\mathbf{w}} dx ds \\ &\leq \frac{\nu}{2} \iint_{Q_t} |\tilde{\mathbf{b}}_x|^2 dx ds + C \int_0^t D(s) \mathbb{H}(s) ds. \end{aligned}$$

Thus, the claim (2.68) is proved.

Adding the above five inequalities and taking a small $\epsilon > 0$, we complete the proof of Theorem 1.1(iii) by Gronwall's inequality.

Thus, the proof to Theorem 1.1 is complete.

3 Proof of Theorem 1.3

Lemma 3.1. *Let (1.4), (1.8) and (1.11) hold. Then $\bar{\mathbf{b}} = \bar{\mathbf{w}} = 0$. Moreover,*

$$\sup_{0 < t < T} \int_{\Omega} (|\mathbf{b}|^2 + |\mathbf{w}|^2) dx + \iint_{Q_T} |\mathbf{b}_x|^2 dx dt \leq C\sqrt{\mu}.$$

Proof. From Theorem 1.1(iii), it suffices to show that $\bar{\mathbf{b}} = \bar{\mathbf{w}} = 0$. To this end, multiplying the equations (1.10)₃ and (1.10)₄ by $\bar{\mathbf{w}}$ and $\bar{\mathbf{b}}$, respectively, and integrating over Q_t , we have

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \bar{\rho} |\bar{\mathbf{w}}|^2 dx - \iint_{Q_t} \bar{\mathbf{b}}_x \cdot \bar{\mathbf{w}} dx ds = 0, \\ & \frac{1}{2} \int_{\Omega} |\bar{\mathbf{b}}|^2 dx + \nu \iint_{Q_t} |\bar{\mathbf{b}}_x|^2 dx ds + \iint_{Q_t} \bar{\mathbf{b}}_x \cdot \bar{\mathbf{w}} dx ds + \frac{1}{2} \iint_{Q_t} \bar{u}_x |\bar{\mathbf{b}}|^2 dx ds = 0. \end{aligned}$$

Adding the two equations yields

$$\frac{1}{2} \int_{\Omega} (\bar{\rho} |\bar{\mathbf{w}}|^2 + |\bar{\mathbf{b}}|^2) dx + \nu \iint_{Q_t} |\bar{\mathbf{b}}_x|^2 dx dt \leq \frac{1}{2} \int_0^t \|\bar{u}_x\|_{L^\infty(\Omega)} \int_{\Omega} |\bar{\mathbf{b}}|^2 dx ds,$$

which together with Gronwall's inequality completes the proof. □

Lemma 3.2. *Let (1.4), (1.8) and (1.11) hold. Then*

$$\sup_{0 < t < T} \int_{\Omega} |\mathbf{w}_x|^2 \omega^2 dx \leq C\sqrt{\mu}.$$

Proof. Note that $\mathbf{w}_0 = \mathbf{b}_0 \equiv 0$. By means of Lemmas 2.13 and 3.1, similar arguments as in Lemmas 2.7 and 2.8 give

$$\begin{aligned} & \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega^2 dx ds \leq C\sqrt{\mu} + C \iint_{Q_t} |\mathbf{w}_x|^2 \omega^2 dx ds, \\ & \int_{\Omega} |\mathbf{w}_x|^2 \omega^2 dx \leq C\sqrt{\mu} + C \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega^2 dx ds, \end{aligned}$$

so

$$\int_{\Omega} |\mathbf{w}_x|^2 \omega^2 dx \leq C\sqrt{\mu} + C \iint_{Q_t} |\mathbf{w}_x|^2 \omega^2 dx ds,$$

and then, Gronwall's inequality yields the desired result. The proof is complete. \square

Lemma 3.3. *Let (1.4), (1.8) and (1.11) hold. Then*

$$\iint_{Q_T} |\mathbf{w}_t|^2 dx dt \leq C\sqrt{\mu}.$$

Proof. Using Lemmas 2.10, 2.13, 3.1 and 3.2 and noticing (2.21), we derive from (2.16) that

$$\begin{aligned} \iint_{Q_T} |\mathbf{w}_t|^2 dx dt &\leq C\mu^2 \iint_{Q_T} |\mathbf{w}_{xx}|^2 dx dt + C \iint_{Q_T} |\mathbf{b}_x|^2 dx dt + C \iint_{Q_T} u^2 |\mathbf{w}_x|^2 dx dt \\ &\leq C\sqrt{\mu} + \int_0^T \|u_x\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\mathbf{w}_x|^2 \omega^2 dx dt \leq C\sqrt{\mu}. \end{aligned}$$

The proof is complete. \square

Lemma 3.4. *Let (1.4), (1.8) and (1.11) hold. Then*

$$\sup_{0 < t < T} \int_{\Omega} |\mathbf{b}_x|^2 dx + \iint_{Q_T} |\mathbf{b}_t|^2 dx dt \leq C\sqrt{\mu}.$$

Proof. Multiplying (1.1)₄ by \mathbf{b}_t , integrating over Q_t and noticing $\mathbf{w}_0 = 0$, we have

$$\frac{\nu}{2} \int_{\Omega} |\mathbf{b}_x|^2 dx + \iint_{Q_t} |\mathbf{b}_t|^2 dx dt = \iint_{Q_t} \mathbf{w}_x \cdot \mathbf{b}_t dx dt - \iint_{Q_t} (u\mathbf{b})_x \cdot \mathbf{b}_t dx dt. \quad (3.1)$$

Using Lemmas 2.10, 3.1 and 3.3, we obtain by the similar arguments as in (2.29) and (2.30)

$$\iint_{Q_t} \mathbf{w}_x \cdot \mathbf{b}_t dx dt \leq C\sqrt{\mu} + \frac{\nu}{4} \int_{\Omega} |\mathbf{b}_x|^2 dx,$$

and

$$- \iint_{Q_t} (u\mathbf{b})_x \cdot \mathbf{b}_t dx dt \leq C\sqrt{\mu} + \frac{1}{2} \iint_{Q_t} |\mathbf{b}_t|^2 dx dt.$$

Substituting them into (3.1), we complete the proof. \square

Now we can prove Theorem 1.3.

Proof of Theorem 1.3 By Theorem 1.1 (ii)-(iii) and Lemma 3.1, one sees that there exists a unique solution $(\bar{\rho}, \bar{u}, \mathbf{0}, \mathbf{0}, \bar{\theta})$ for the limit problem (1.10) in \mathbb{F} .

Next, we are ready to show the second part of this theorem. Denote $\omega_\delta : [0, 1] \rightarrow [0, 1]$ for $\delta \in (0, 1/2)$ by

$$\omega_\delta(x) = \begin{cases} x, & 0 \leq x \leq \delta, \\ \delta, & \delta \leq x \leq 1 - \delta, \\ 1 - x, & 1 - \delta \leq x \leq 1. \end{cases}$$

Multiplying (2.16) by $\mathbf{w}_{xx}\omega_\delta^n(x)$ ($n = 1, 2, \dots$) and integrating over Q_t , we have

$$\begin{aligned} \mu \iint_{Q_t} |\mathbf{w}_{xx}|^2 \frac{\omega_\delta^n}{\rho} dx ds &= \iint_{Q_t} \mathbf{w}_t \cdot \mathbf{w}_{xx} \omega_\delta^n dx dt + \iint_{Q_t} u \mathbf{w}_x \cdot \mathbf{w}_{xx} \omega_\delta^n dx ds \\ &\quad - \iint_{Q_t} \mathbf{b}_x \cdot \mathbf{w}_{xx} \frac{\omega_\delta^n}{\rho} dx ds. \end{aligned} \quad (3.2)$$

Integrating by parts, using (2.16) and noticing $\mathbf{w}_0 = 0$, we have

$$\begin{aligned} &\iint_{Q_t} \mathbf{w}_t \cdot \mathbf{w}_{xx} \omega_\delta^n dx dt \\ &= -\frac{1}{2} \int_\Omega |\mathbf{w}_x|^2 \omega_\delta^n dx - n \iint_{Q_t} \mathbf{w}_t \cdot \mathbf{w}_x \omega_\delta^{n-1} \omega'_\delta dx dt \\ &= -\frac{1}{2} \int_\Omega |\mathbf{w}_x|^2 \omega_\delta^n dx - n \iint_{Q_t} \left(\frac{\mu}{\rho} \mathbf{w}_{xx} - u \mathbf{w}_x + \frac{\mathbf{b}_x}{\rho} \right) \cdot \mathbf{w}_x \omega_\delta^{n-1} \omega'_\delta dx ds \\ &\leq -\frac{1}{2} \int_\Omega |\mathbf{w}_x|^2 \omega_\delta^n dx + \frac{\mu}{2} \iint_{Q_t} |\mathbf{w}_{xx}|^2 \frac{\omega_\delta^n}{\rho} dx ds + C_n \mu \iint_{Q_t} |\mathbf{w}_x|^2 \omega_\delta^{n-2} dx ds \\ &\quad + C_n \iint_{Q_t} |u| |\mathbf{w}_x|^2 \omega_\delta^{n-1} |\omega'_\delta| dx ds - n \iint_{Q_t} \mathbf{b}_x \cdot \mathbf{w}_x \frac{\omega_\delta^{n-1} \omega'_\delta}{\rho} dx ds, \quad n = 2, 3, \dots \end{aligned} \quad (3.3)$$

Here and in what follows, C and C_n are positive constants independent of μ and δ .

By the mean value theorem and $u(1, t) = u(0, t) = 0$, we have

$$|u(x, t)| \leq \|u_x\|_{L^\infty(\Omega)} \omega_\delta(x), \quad \forall x \in [0, \delta] \cup [1 - \delta, 1], \quad (3.4)$$

which together with the definition of ω_δ gives

$$\begin{aligned} \iint_{Q_t} |u| |\mathbf{w}_x|^2 \omega_\delta^{n-1} |\omega'_\delta| dx ds &= \int_0^t \int_0^\delta |u| |\mathbf{w}_x|^2 \omega_\delta^{n-1} dx ds + \int_0^t \int_{1-\delta}^1 |u| |\mathbf{w}_x|^2 \omega_\delta^{n-1} dx ds \\ &\leq \int_0^t \|u_x\|_{L^\infty(\Omega)} \int_\Omega |\mathbf{w}_x|^2 \omega_\delta^n dx ds, \end{aligned}$$

thus

$$\begin{aligned} &\iint_{Q_t} \mathbf{w}_t \cdot \mathbf{w}_{xx} \omega_\delta^n dx dt \\ &\leq -\frac{1}{2} \int_\Omega |\mathbf{w}_x|^2 \omega_\delta^n dx + \frac{\mu}{2} \iint_{Q_t} |\mathbf{w}_{xx}|^2 \frac{\omega_\delta^n}{\rho} dx ds + C_n \mu \iint_{Q_t} |\mathbf{w}_x|^2 \omega_\delta^{n-2} dx ds \\ &\quad + C \int_0^t \|u_x\|_{L^\infty(\Omega)} \int_\Omega |\mathbf{w}_x|^2 \omega_\delta^n dx ds - n \iint_{Q_t} \mathbf{b}_x \cdot \mathbf{w}_x \frac{\omega_\delta^{n-1} \omega'_\delta}{\rho} dx ds. \end{aligned}$$

To estimate the second integral on the right-hand side of (3.2), we have by integrating by parts and noticing (3.4)

$$\begin{aligned} \iint_{Q_t} u \mathbf{w}_x \cdot \mathbf{w}_{xx} \omega_\delta^n dx ds &= -\frac{1}{2} \iint_{Q_t} |\mathbf{w}_x|^2 [u_x \omega_\delta^n + n u \omega_\delta^{n-1} \omega'_\delta] dx ds \\ &\leq C_n \int_0^t \|u_x\|_{L^\infty(\Omega)} \int_\Omega |\mathbf{w}_x|^2 \omega_\delta^n dx ds. \end{aligned}$$

As to the third term on the right-hand side of (3.2), we have

$$\begin{aligned}
& - \iint_{Q_t} \mathbf{b}_x \cdot \mathbf{w}_{xx} \frac{\omega_\delta^n}{\rho} dx ds \\
& = \iint_{Q_t} \mathbf{w}_x \cdot \mathbf{b}_{xx} \frac{\omega_\delta^n}{\rho} dx ds - \iint_{Q_t} \mathbf{w}_x \cdot \mathbf{b}_x \frac{\omega_\delta^n \rho_x}{\rho^2} dx ds + n \iint_{Q_t} \mathbf{b}_x \cdot \mathbf{w}_x \frac{\omega_\delta^{n-1} \omega'_\delta}{\rho} dx ds \\
& \leq C \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega_\delta^n dx ds + C \iint_{Q_t} |\mathbf{w}_x|^2 \omega_\delta^n dx ds + C \iint_{Q_t} |\mathbf{b}_x|^2 \omega_\delta^n \rho_x^2 dx ds \\
& \quad + n \iint_{Q_t} \mathbf{b}_x \cdot \mathbf{w}_x \frac{\omega_\delta^{n-1} \omega'_\delta}{\rho} dx ds \\
& \leq C \sqrt{\mu} \delta^{n-1} + C \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega_\delta^n dx ds + C \iint_{Q_t} |\mathbf{w}_x|^2 \omega_\delta^n dx ds \\
& \quad + n \iint_{Q_t} \mathbf{b}_x \cdot \mathbf{w}_x \frac{\omega_\delta^{n-1} \omega'_\delta}{\rho} dx ds,
\end{aligned}$$

where we used the fact by (2.10)₁, Lemma 3.1 and $0 \leq \omega_\delta(x) \leq \delta$

$$\begin{aligned}
\iint_{Q_t} |\mathbf{b}_x|^2 \omega_\delta^n \rho_x^2 dx ds & \leq C \int_0^t \| |\mathbf{b}_x|^2 \omega_\delta^n \|_{L^\infty(\Omega)} ds \leq C \iint_{Q_t} |(|\mathbf{b}_x|^2 \omega_\delta^n)_x| dx ds \\
& \leq C_n \iint_{Q_t} |\mathbf{b}_x|^2 |\omega_\delta^{n-1} \omega'_\delta| dx ds + C \iint_{Q_t} |\mathbf{b}_x \cdot \mathbf{b}_{xx}| \omega_\delta^n dx ds \\
& \leq C_n \sqrt{\mu} \delta^{n-1} + C \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega_\delta^n dx ds.
\end{aligned}$$

Substituting the above results into (3.2) yields

$$\begin{aligned}
& \int_\Omega |\mathbf{w}_x|^2 \omega_\delta^n dx + \mu \iint_{Q_t} |\mathbf{w}_{xx}|^2 \omega_\delta^n dx ds \\
& \leq C_n \sqrt{\mu} \delta^{n-1} + C_n \mu \iint_{Q_t} |\mathbf{w}_x|^2 \omega_\delta^{n-2} dx ds \\
& \quad + C \int_0^t [1 + \|u_x\|_{L^\infty(\Omega)}] \int_\Omega |\mathbf{w}_x|^2 \omega_\delta^n dx ds + C \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega_\delta^n dx ds.
\end{aligned} \tag{3.5}$$

It remains to treat the relation between the terms $\iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega_\delta^n dx ds$ and $\iint_{Q_t} |\mathbf{w}_x|^2 \omega_\delta^n dx ds$. To this end, we multiply (1.1)₄ by $\mathbf{b}_{xx} \omega_\delta^n(x)$ ($n = 2, 3, \dots$) and integrate over Q_t to obtain

$$\begin{aligned}
\nu \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega_\delta^n dx ds & = \iint_{Q_t} \mathbf{b}_t \cdot \mathbf{b}_{xx} \omega_\delta^n dx dt + \iint_{Q_t} (u \mathbf{b})_x \cdot \mathbf{b}_{xx} \omega_\delta^n dx ds \\
& \quad - \iint_{Q_t} \mathbf{w}_x \cdot \mathbf{b}_{xx} \omega_\delta^n dx ds.
\end{aligned} \tag{3.6}$$

To estimate the first term on right-hand side of (3.6), we use Young's inequality, Lemma 3.4 and $0 \leq \omega_\delta(x) \leq \delta$ to obtain

$$\iint_{Q_t} \mathbf{b}_t \cdot \mathbf{b}_{xx} \omega_\delta^n dx dt \leq C \sqrt{\mu} \delta^n + \frac{\nu}{4} \iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega_\delta^n dx dt.$$

Next we deal with the second term on right-hand side of (3.6). By $0 \leq \omega_\delta(x) \leq \delta$ and Lemmas 2.10 and 3.1, we have

$$\begin{aligned} \iint_{Q_t} |(u\mathbf{b})_x|^2 \omega_\delta^n dx ds &\leq C \iint_{Q_t} u^2 |\mathbf{b}_x|^2 \omega_\delta^n dx ds + C \iint_{Q_t} u_x^2 |\mathbf{b}|^2 \omega_\delta^n dx ds \\ &\leq C \sqrt{\mu} \delta^n + C \delta^n \int_0^t \|u_x\|_{L^\infty(\Omega)}^2 \int_\Omega |\mathbf{b}|^2 dx ds \leq C \sqrt{\mu} \delta^n. \end{aligned}$$

As to the third term on right-hand side of (3.6), we have by Young's inequality

$$- \iint_{Q_t} \mathbf{w}_x \cdot \mathbf{b}_{xx} \omega_\delta^n dx ds \leq C \iint_{Q_t} |\mathbf{w}_x| \omega_\delta^n dx ds + \frac{\nu}{4} \iint_{Q_t} |\mathbf{b}_{xx}| \omega_\delta^n dx ds.$$

Substituting them into (3.6) yields

$$\iint_{Q_t} |\mathbf{b}_{xx}|^2 \omega_\delta^n dx ds \leq C \sqrt{\mu} \delta^n + C \iint_{Q_t} |\mathbf{w}_x|^2 \omega_\delta^n dx ds, \quad (3.7)$$

and inserting it into (3.5) and using Gronwall's inequality, we obtain the iteration

$$\int_\Omega |\mathbf{w}_x|^2 \omega_\delta^n dx \leq C_n \sqrt{\mu} \delta^{n-1} + C_n \mu \iint_{Q_t} |\mathbf{w}_x|^2 \omega_\delta^{n-2} dx ds, \quad n = 2, 3, \dots \quad (3.8)$$

Note that the above results still hold for $n = 1$. Since the term $-\iint_{Q_t} \frac{\mu}{\rho} \mathbf{w}_{xx} \cdot \mathbf{w}_x \omega'_\delta dx ds$ in the equality of (3.3) with $n = 1$ can be dealt with as follows

$$- \iint_{Q_t} \frac{\mu}{\rho} \mathbf{w}_{xx} \cdot \mathbf{w}_x \omega'_\delta dx ds \leq C \sqrt{\mu} \iint_{Q_t} |\mathbf{w}_x|^2 dx ds + C \mu^{3/2} \iint_{Q_t} |\mathbf{w}_{xx}|^2 dx ds \leq C,$$

where we used Lemma 2.13, a similar argument as above gives, instead of (3.8),

$$\int_\Omega |\mathbf{w}_x|^2 \omega_\delta dx \leq C. \quad (3.9)$$

So, we derive from (3.8) that

$$\begin{aligned} \int_\Omega |\mathbf{w}_x|^2 \omega_\delta^2 dx &\leq C(\sqrt{\mu} \delta + \sqrt{\mu}), \\ \int_\Omega |\mathbf{w}_x|^2 \omega_\delta^3 dx &\leq C(\sqrt{\mu} \delta^2 + \mu). \end{aligned}$$

Next, taking $n = 4, 5, 6, 7$ in (3.8), respectively, we get

$$\begin{aligned} \int_\Omega |\mathbf{w}_x|^2 \omega_\delta^4 dx &\leq C(\sqrt{\mu} \delta^3 + \mu^{3/2} \delta + \mu^{3/2}), \\ \int_\Omega |\mathbf{w}_x|^2 \omega_\delta^5 dx &\leq C(\sqrt{\mu} \delta^4 + \mu^{3/2} \delta^2 + \mu^2), \\ \int_\Omega |\mathbf{w}_x|^2 \omega_\delta^6 dx &\leq C(\sqrt{\mu} \delta^5 + \mu^{3/2} \delta^3 + \mu^{5/2} \delta + \mu^{5/2}), \\ \int_\Omega |\mathbf{w}_x|^2 \omega_\delta^7 dx &\leq C(\sqrt{\mu} \delta^6 + \mu^{3/2} \delta^4 + \mu^{5/2} \delta^2 + \mu^3), \end{aligned}$$

thus, an induction gives

$$\int_{\Omega} |\mathbf{w}_x|^2 \omega_{\delta}^n dx \leq \begin{cases} C_n (\sqrt{\mu} \delta^{n-1} + \mu^{3/2} \delta^{n-3} + \dots + \mu^{(n-2)/2} \delta^2 + \mu^{(n-1)/2}) & (n = \text{odd}), \\ C_n (\sqrt{\mu} \delta^{n-1} + \mu^{3/2} \delta^{n-3} + \dots + \mu^{(n-1)/2} \delta + \mu^{(n-1)/2}) & (n = \text{even}), \end{cases} \quad (3.10)$$

where $n = 2, 3, \dots$, which together with the definition of ω_{δ} gives

$$\int_{\delta}^{1-\delta} |\mathbf{w}_x|^2 dx \leq \begin{cases} C_n (\tau + \tau^3 + \dots + \tau^{n-2}) + C_n \mu^{(n-1)/2} / \delta^n & (n = \text{odd}), \\ C_n (\tau + \tau^3 + \dots + \tau^{n-1}) + C_n \mu^{(n-1)/2} / \delta^n & (n = \text{even}), \end{cases} \quad (3.11)$$

where $\delta \in (0, 1/2)$, $\tau = \sqrt{\mu}/\delta$.

On the other hand, we have by the mean value theorem and Lemma 3.1

$$\begin{aligned} \|\mathbf{w}\|_{L^{\infty}(\delta, 1-\delta)} &\leq \frac{1}{1-2\delta} \int_{\delta}^{1-\delta} |\mathbf{w}| dx + \int_{\delta}^{1-\delta} |\mathbf{w}_x| dx \\ &\leq C \mu^{1/4} + \left(\int_{\delta}^{1-\delta} |\mathbf{w}_x|^2 dx \right)^{1/2}, \quad \forall \delta \in (0, 1/4), \end{aligned}$$

which together with (3.11) implies that any function $\delta(\mu)$ with $\delta(\mu) \downarrow 0$ and $\frac{\mu^{(n-1)/(2n)}}{\delta(\mu)} = \frac{\mu^{1/2-1/(2n)}}{\delta(\mu)} \rightarrow 0$ as $\mu \rightarrow 0$ satisfies

$$\lim_{\mu \rightarrow 0} \|\mathbf{w}\|_{L^{\infty}(0,T;L^{\infty}(\delta(\mu), 1-\delta(\mu)))} = 0. \quad (3.12)$$

Since n can be arbitrarily large, we see that $\delta(\mu) = \mu^{\alpha}$ satisfies (3.12) for any $\alpha \in (0, 1/2)$. The proof is completed.

Acknowledgments

The authors would like to thank Professor Tong Yang at City University of Hong Kong for valuable discussions on Lemma 2.14 during their visit to City University of Hong Kong. The research was supported in part by the NSFC (grants 11571062, 11571380, 11401078), Guangdong Natural Science Foundation (grant 2014A030313161), the Program for Liaoning Excellent Talents in University (grant LJQ2013124) and the Fundamental Research Fund for the Central Universities (grant DC201502050202).

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